Attractive Centers I

Chests, Hearts, and Symmetry–Displacement Geometry

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Abstract

Let $P = (p_1, \dots, p_N)$ be a finite configuration of points in Euclidean space. We study center rules satisfying an order-free monotonicity principle: if one input point is displaced by h, the center is not allowed to move against that displacement,

$$h \cdot \big(X(P_{i,h}) - X(P)\big) \geq 0.$$

This is the single-point consistent monotonicity (SC-monotonicity) of multidimensional aggregation theory. The present problem is geometric: for a *fixed* configuration, determine the region containing every invariant attractive center and the region actually attainable by globally admissible center rules.

The theory begins with an exact one-dimensional model. For ordered real data $x_{(1)} \leq \cdots \leq x_{(N)}$, put

$$q_k = \frac{x_{(k)} + x_{(N+1-k)}}{2}, \qquad L = \min_k q_k, \quad U = \max_k q_k.$$

We prove that the exact attainable locus of invariant attractive means is the interval $[L, U]$. For three reals this is precisely the interval between the median and the midrange. For point sets increasingly uniformly filling $[a, b]$, this entire interval converges to the single midpoint $(a + b)/2$, giving a first exact stability theorem for the large-sample programme.

In higher dimensions a *chest* is a certified convex outer region containing every attractive center value, whereas a *heart* is a convex inner region whose points are realized by center maps in a specified regularity class. Euclidean equivariance and attractivity force every center into the convex hull. More generally, a symmetry-fixed affine subspace together with vertex displacements produces linear support inequalities. Finite well-founded networks of such moves give polyhedral chests after projection of the intermediate center variables; linear programming detects redundant inequalities and Farkas duality gives algebraic certificates.

For three noncollinear planar points we derive the standard chest

$$\mathcal{C}_{\mathrm{std}}(ABC) = \mathrm{conv}\{A, B, C\} \cap \mathrm{CP}(ABC) \cap \mathrm{CNT}(ABC),$$

where CP is the characteristic parallelogram and CNT is the *internal* characteristic Napoleon triangle. We give corrected proofs of both constructions and conjecture that this standard chest is already the exact outer region obtainable from the full finite symmetry–displacement certificate calculus. Four-point examples show why intermediate center variables are logically essential. A fully specified four-point example is included: its elementary strip chest occupies 58.4105% of the hull, useful pair-bisectors reduce this to 21.2397%, and an independent two-order network cuts the combined rigorous chest to 13.4127%. Finally we formulate a distributional notion of stability for point clouds converging at constant density to a Euclidean region.

Contents

1 Introduction

The project originates in the theory of averages and means of real numbers. On the real line, symmetry, translation and scale behaviour, continuity, and coordinatewise monotonicity are natural axioms. Sýkora distinguishes an *average*, for which the usual order bound may be dropped, from a *mean*, which is constrained by the input range; a continuous monotone mean is called regular [7]. The structural axioms of an average, unlike the total order of $\mathbb{R}$, make sense without change for complex numbers and vectors.

This observation provides the original bridge to triangle geometry. Three complex numbers are three points in the Euclidean plane, and a classical triangle center can be viewed as an invariant average of those three vectors. The question "which regular means are possible?" therefore has a geometric continuation: if one vertex is moved, what positive-response law should replace ordinary monotonicity, and where can every such center lie?

For a center rule X, the response law used throughout the paper is

$$h \cdot \big[X(p_1, \dots, p_i + h, \dots, p_N) - X(P)\big] \geq 0. \tag{1}$$

Pérez-Fernández, De Baets, and Gagolewski call this single-point consistent monotonicity (SC-monotonicity) [6]. We use the term *attractivity*: the center may move sideways or in the same general direction as the moved datum, but not against it.

The geometric problem is two-sided. For one fixed configuration P, ask

(1) what region must contain $X(P)$ for *every* invariant attractive center rule X?

(2) what points are actually attained as $X(P)$ by globally admissible center rules in a chosen regularity class?

The first is an outer problem and leads to *chests*; the second is an inner realization problem and leads to *hearts*. Chests can be derived from finite inequalities and need no differentiability or continuity. Hearts are more restrictive: to certify a smooth heart point one must construct an entire center function satisfying attractivity on its global domain.

Before passing to vectors, however, the one-dimensional problem should be solved exactly. It is not merely motivational: it is the prototype in which the chest and heart coincide, the finite symmetry–displacement method is complete, and the large-sample stability sought later can already be proved.

2 The exact one-dimensional prototype

Let $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, and write its order statistics as

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(N)}.$$

For $1 \leq k \leq N$, define the symmetric order-statistic averages

$$Q_k(x) = \frac{x_{(k)} + x_{(N+1-k)}}{2}. \tag{2}$$

For the fixed data vector x, write

$$L(x) = \min_{1 \leq k \leq N} Q_k(x), \qquad U(x) = \max_{1 \leq k \leq N} Q_k(x). \tag{3}$$

In one dimension, (1) is exactly coordinatewise monotonicity: if one input is increased, the output cannot decrease.

Theorem 2.1 (Exact one-dimensional chest–heart theorem)**.** *Let $M : \mathbb{R}^N \to \mathbb{R}$ be permutation invariant, equivariant under Euclidean isometries and positive changes of scale, and attractive in the sense of* (1)*. Then for every $x \in \mathbb{R}^N$,*

$$\boxed{L(x) \le M(x) \le U(x).} \tag{4}$$

Conversely, every point of $[L(x), U(x)]$ is attained at x by a continuous invariant attractive center rule. Hence for the full continuous invariant attractive class the exact attainable locus is

$$\boxed{\mathcal{V}_N(x) = [L(x), U(x)].}$$

Proof. By permutation invariance we may relabel the input so that $x_1 \le \cdots \le x_N$. Put $L = L(x)$. For every paired index $k < N + 1 - k$, the definition of L gives

$$2L \le x_k + x_{N+1-k}.$$

Construct a new labelled configuration y by

$$y_k = 2L - x_{N+1-k}, \qquad y_{N+1-k} = x_{N+1-k}$$

for one representative of each pair. If N is odd, put $y_{(N+1)/2} = L$. Then $y_i \le x_i$ coordinatewise, and the multiset $\{y_i\}$ is invariant under reflection $r_L(t) = 2L - t$. Euclidean equivariance and permutation invariance therefore force

$$M(y) = r_L(M(y)),$$

so $M(y) = L$. Coordinatewise monotonicity now gives $M(x) \ge L$.

Similarly, put $U = U(x)$. Since

$$x_k + x_{N+1-k} \le 2U,$$

construct z by

$$z_k = x_k, \qquad z_{N+1-k} = 2U - x_k,$$

and, in the odd case, $z_{(N+1)/2} = U$. Then $z_i \ge x_i$, the multiset $\{z_i\}$ is reflection-symmetric about U, and hence $M(z) = U$. Monotonicity gives $M(x) \le U$. This proves (4) without any continuity assumption.

For exact attainability, every map Q_k in (2) is continuous, permutation invariant, equivariant under translations, reflections, and positive scaling, and coordinatewise nondecreasing because each order statistic is coordinatewise nondecreasing. Choose indices k_- and k_+ with

$$Q_{k_-}(x) = L(x), \qquad Q_{k_+}(x) = U(x).$$

For $0 \le t \le 1$, the convex mixture

$$M_t = (1 - t)Q_{k_-} + tQ_{k_+}$$

has the same properties and satisfies

$$M_t(x) = (1 - t)L(x) + tU(x).$$

Thus every point of the interval is attained. □

Corollary 2.2 (Three real inputs)**.** *For $x_{(1)} \le x_{(2)} \le x_{(3)}$, the exact attainable interval is the interval whose endpoints are the median and the midrange:*

$$\boxed{\mathcal{V}_3(x) = \left[\min\left(x_{(2)}, \frac{x_{(1)} + x_{(3)}}{2}\right), \max\left(x_{(2)}, \frac{x_{(1)} + x_{(3)}}{2}\right)\right].}$$

Proof. Here $Q_1 = Q_3 = (x_{(1)} + x_{(3)})/2$ and $Q_2 = x_{(2)}$. $\square$

The proof already has the geometry used later. A reflection-symmetric predecessor forces the center to a known fixed set, and monotone restoration of the data produces a support inequality. In one dimension two such certificates are sufficient, and explicit attractive means fill the entire outer interval. Thus the finite certificate calculus introduced below is complete in this prototype case.

2.1 Uniform filling of an interval

The exact formula also gives a first rigorous large-sample stability theorem. Let $a < b$, and for each N let

$$x_{(1)}^{(N)} \le \cdots \le x_{(N)}^{(N)}$$

be points in $[a, b]$. Compare them with the uniform reference quantiles

$$r_i^{(N)} = a + \frac{i}{N+1}(b-a), \qquad 1 \le i \le N,$$

and define

$$\delta_N = \max_i \left| x_{(i)}^{(N)} - r_i^{(N)} \right|. \tag{5}$$

Corollary 2.3 (One-dimensional uniform stability)**.** *If $\delta_N \to 0$, then the entire exact attainable locus converges in Hausdorff distance to the midpoint of the interval:*

$$\boxed{d_H\left(\mathcal{V}_N(x^{(N)}), \left\{\frac{a+b}{2}\right\}\right) \le \delta_N \longrightarrow 0.}$$

In particular, this holds almost surely for independent samples from the uniform distribution on $[a, b]$.

Proof. The reference points satisfy

$$r_i^{(N)} + r_{N+1-i}^{(N)} = a + b.$$

Therefore for every k,

$$\left| Q_k(x^{(N)}) - \frac{a+b}{2} \right| \le \frac{1}{2}\left(|x_{(k)}^{(N)} - r_k^{(N)}| + |x_{(N+1-k)}^{(N)} - r_{N+1-k}^{(N)}| \right) \le \delta_N.$$

By Theorem 2.1, the attainable locus is the interval between the minimum and maximum of these Q_k's, proving the Hausdorff bound. For an i.i.d. uniform sample, $\delta_N \to 0$ almost surely by the standard uniform convergence of the empirical distribution and its order statistics [8]. $\square$

This result captures the intended notion of stability in its simplest form: as a random or deterministic cloud fills a region at constant density, not merely one preferred mean but the *entire admissible centrality region* can collapse to a canonical limit. Later sections formulate the corresponding question for Euclidean regions.

3 Invariant attractive centers and regularity classes

Let

$$\mathcal{C}_{N,d} = (\mathbb{R}^d)^N$$

be the space of labelled N-point configurations. Labels are used only to state vertexwise response conditions; permutation invariance makes the output depend only on the multiset of points.

Definition 3.1 (Invariant center rule)**.** A map $X : \mathcal{C}_{N,d} \to \mathbb{R}^d$ is an *invariant center rule* if

(C1) X is permutation invariant;

(C2) $X(Qp_1 + t, \dots, Qp_N + t) = QX(P) + t$ for every orthogonal Q and translation t;

(C3) $X(sp_1, \dots, sp_N) = sX(P)$ for every $s > 0$.

No continuity or differentiability is included in this definition.

Definition 3.2 (Attractivity)**.** An invariant center rule is *attractive* if (1) holds for every configuration P, every index i, and every displacement $h \in \mathbb{R}^d$.

We write $\mathcal{A}^{\mathrm{fr}}_{N,d}$ for the full finite-response attractive class. On a chosen domain we use the conventional regularity notation

$$\mathcal{A}^{\omega}_{N,d} \subseteq \mathcal{A}^{\infty}_{N,d} \subseteq \mathcal{A}^{1}_{N,d} \subseteq \mathcal{A}^{0}_{N,d} \subseteq \mathcal{A}^{\mathrm{fr}}_{N,d},$$

where $\mathcal{A}^0$ is continuous, $\mathcal{A}^1$ is C^1, and the other superscripts have their usual meanings. The smoothness requirement applies to the stated regular domain, not to the total collision; a homogeneous nonlinear rule need not be differentiable there. For triangle hearts we require continuous extension to all configurations and the specified regularity only on nondegenerate triangles. Chests in this paper are usually proved for the broad class $\mathcal{A}^{\mathrm{fr}}$; hearts may be attached to any one of the regularity subclasses.

Two familiar properties of means are automatic consequences of the invariance axioms.

Proposition 3.3 (Idempotence and the two-point center)**.** *Every invariant center rule satisfies*

$$X(p, \dots, p) = p.$$

For $N = 2$, in every Euclidean dimension, the unique invariant center rule is

$$\boxed{X(A, B) = \frac{A + B}{2}.}$$

No attractivity assumption is needed for either statement.

Proof. Positive scaling gives $X(0, \dots, 0) = sX(0, \dots, 0)$ for every $s > 0$, hence $X(0, \dots, 0) = 0$. Translation equivariance then gives idempotence.

For two points let $m = (A + B)/2$. The central reflection $r(x) = 2m - x$ exchanges A and B. Permutation invariance and Euclidean equivariance imply $X(A, B) = r(X(A, B))$, whose only solution is m. □

Thus the first nontrivial cases occur at arity three: an interval for three real points and a genuinely planar region for three noncollinear planar points.

3.1 The infinitesimal test

For C^1 center rules attractivity has the semidefinite form used in later heart calculations.

Proposition 3.4 (Jacobian criterion)**.** *Let $X \in C^1(\Omega, \mathbb{R}^d)$, where Ω is closed under the one-vertex line segments under consideration. Put*

$$J_i(P) = D_{p_i} X(P), \qquad S_i(P) = \frac{J_i(P) + J_i(P)^T}{2}.$$

Then X is attractive on Ω if and only if

$$S_i(P) \succeq 0$$

for every $P \in \Omega$ and every i.

Proof. For an infinitesimal displacement th,

$$X(P_{i,th}) - X(P) = tJ_i(P)h + o(t),$$

so finite attractivity implies $h^T J_i(P)h = h^T S_i(P)h \geq 0$. Conversely, along $P(t) = P_{i,th}$,

$$h \cdot [X(P_{i,h}) - X(P)] = \int_0^1 h^T J_i(P(t))h \, \mathrm{d}t = \int_0^1 h^T S_i(P(t))h \, \mathrm{d}t \geq 0.$$

If a relevant segment meets an excluded degenerate stratum, the same conclusion follows when the rule and the finite inequality extend continuously across that stratum. □

Translation, scaling, and rotational equivariance imply, wherever the Jacobians exist,

$$\sum_{i=1}^{N} J_i = I_d, \tag{6}$$

$$\sum_{i=1}^{N} J_i p_i = X(P), \tag{7}$$

$$\sum_{i=1}^{N} J_i \Omega p_i = \Omega X(P) \qquad (\Omega^T = -\Omega). \tag{8}$$

These identities are useful for smooth-heart computations but are not needed for the finite chest theory.

4 Convex-hull internality is a theorem

A center is often required axiomatically to belong to the convex hull of the data. For attractive Euclidean-equivariant centers this is unnecessary.

Theorem 4.1 (Convex-hull theorem)**.** *Every Euclidean-equivariant attractive center rule $X : \mathcal{C}_{N,d} \to \mathbb{R}^d$ satisfies*

$$X(P) \in \operatorname{conv}\{p_1, \ldots, p_N\}$$

for every finite configuration P. The same statement holds for a rule defined only on a full-dimensional regular domain if it extends continuously to the projected lower-dimensional configurations used in the proof.

Proof. Let

$$H = \{z : n \cdot z = \alpha\}, \qquad H^+ = \{z : n \cdot z \geq \alpha\}, \qquad \|n\| = 1,$$

be a supporting hyperplane and inward half-space containing P. Project every input orthogonally onto H:

$$q_i = p_i - t_i n, \qquad t_i = n \cdot p_i - \alpha \geq 0.$$

Reflection in H fixes $Q = (q_1, \ldots, q_N)$. Euclidean equivariance therefore forces $X(Q)$ to lie in H, so

$$n \cdot X(Q) = \alpha.$$

Restore the points one at a time. The i-th displacement is $t_i n$. Attractivity gives

$$n \cdot (X_i - X_{i-1}) \geq 0$$

whenever $t_i > 0$. Summing yields

$$n \cdot X(P) \geq n \cdot X(Q) = \alpha.$$

Thus $X(P) \in H^+$. Intersecting all supporting half-spaces gives the convex hull. □

Remark 4.2. The proof needs no barycentric positivity, differentiability, or idempotence. It uses only reflection equivariance, the finite displacement inequality, and access to the projected configuration. In this sense internality is a consequence of positive Euclidean response rather than an independent "between-ness" axiom.

5 Attainable loci, chests, and hearts

Fix a class $\mathcal{A}$ of invariant attractive center rules and a target configuration P.

Definition 5.1 (Attainable locus)**.** The *attainable center locus* of $\mathcal{A}$ at P is

$$\mathcal{V}_{\mathcal{A}}(P) = \{X(P) : X \in \mathcal{A}\}.$$

Its closure

$$\overline{\mathcal{V}}_{\mathcal{A}}(P) = \overline{\{X(P) : X \in \mathcal{A}\}}$$

is the *closed attainable locus.*

The two sets are not identified: a limit point of attainable values need not itself be attained by a center rule belonging to $\mathcal{A}$.

Definition 5.2 (Chest and heart)**.** For a fixed P:

- a convex set $C(P)$ is an $\mathcal{A}$*-chest* if $\mathcal{V}_{\mathcal{A}}(P) \subseteq C(P)$;
- a chest is *certified in a proof calculus* $\mathfrak{C}$ if it is the target set of a finite, checkable derivation using only the rules of $\mathfrak{C}$;
- a convex set $H(P)$ is an $\mathcal{A}$*-heart* if $H(P) \subseteq \mathcal{V}_{\mathcal{A}}(P)$;
- a convex set $\overline{H}(P)$ is a *closed heart approximation* if $\overline{H}(P) \subseteq \overline{\mathcal{V}}_{\mathcal{A}}(P)$.

This distinction is essential. "Chest" is a semantic statement about all maps in $\mathcal{A}$; "certified chest" records how that statement was proved. When $\mathcal{A}$ is closed under convex mixing, we call

$$C^{\mathcal{A}}_{\text{exact}}(P) := \mathcal{V}_{\mathcal{A}}(P)$$

the *exact chest* (and its closure the exact closed chest). This name is order-theoretic: it does not assert that the locus has a finite description or that membership can be decided. The optimal-simple-chest problem below is the different, syntactic question of what a specified finite certificate calculus can prove from attractivity and invariance alone.

We use *certified outer chest* for a proved outer bound, *certified inner heart* for a constructed subset, and *exact attainable locus* for the semantic object. For one convex class the maximal heart and minimal chest coincide; a gap between current inner and outer bounds measures incomplete knowledge, not two intrinsically different semantic loci. The same fixed configuration can be studied with several nested regularity classes:

$$\mathcal{V}_{\mathcal{A}^{\omega}}(P) \subseteq \mathcal{V}_{\mathcal{A}^{\infty}}(P) \subseteq \mathcal{V}_{\mathcal{A}^{1}}(P) \subseteq \mathcal{V}_{\mathcal{A}^{0}}(P).$$

Proposition 5.3 (Convex mixing)**.** *If $\mathcal{A}$ is closed under convex combinations of center maps, then $\mathcal{V}_{\mathcal{A}}(P)$ is convex. Consequently it is the unique maximal $\mathcal{A}$-heart and the unique minimal $\mathcal{A}$-chest in the unrestricted semantic sense. Its closure is the unique minimal closed chest containing it.*

Proof. If $X, Y \in \mathcal{A}$ and $0 \le t \le 1$, then $Z = tX + (1-t)Y$ preserves the linear equivariances. For the finite response law,

$$h \cdot [Z(P_{i,h}) - Z(P)] = t\, h \cdot [X(P_{i,h}) - X(P)] + (1-t)\, h \cdot [Y(P_{i,h}) - Y(P)] \ge 0.$$

Thus $Z \in \mathcal{A}$, and $Z(P) = tX(P) + (1-t)Y(P)$. The remaining claims follow from the definitions. $\square$

This proposition explains the role of the two proof directions. Abstractly, the exact attainable locus is the object of interest; practically, it is approached by

$$\boxed{\text{constructed hearts}} \quad \nearrow \quad \mathcal{V}_{\mathcal{A}}(P) \quad \searrow \quad \boxed{\text{certified chests}}.$$

6 The fixed-subspace displacement principle

The elementary engine behind most chest inequalities is symmetry. Let G be a subgroup of Euclidean isometries that preserves a configuration Q as a multiset. Permutation invariance and Euclidean equivariance imply

$$X(Q) \in \mathrm{Fix}(G) = \{x : gx = x \text{ for all } g \in G\}.$$

When nonempty, this fixed set is an affine subspace.

Theorem 6.1 (Fixed-subspace displacement)**.** *Let symmetry force $X(Q) \in L$, where $L \subset \mathbb{R}^d$ is an affine subspace for every invariant center rule. Suppose P is obtained from Q by moving one input through a displacement d, and suppose*

$$d \cdot y = \beta \qquad \text{for every } y \in L.$$

Then every attractive invariant center satisfies

$$d \cdot X(P) \ge \beta. \tag{9}$$

Proof. For the true but unknown predecessor center $y = X(Q) \in L$, attractivity gives

$$d \cdot (X(P) - y) \ge 0.$$

Since $d \cdot y = \beta$, (9) follows. $\square$

Corollary 6.2 (Parallel restoration)**.** *Suppose symmetry forces the predecessor center into the hyperplane*

$$L = \{x : n \cdot x = \alpha\}, \qquad \|n\| = 1.$$

If finitely many inputs are then restored sequentially by displacements $t_j n$ with $t_j \ge 0$, the final center satisfies

$$n \cdot X(P) \ge \alpha.$$

The reverse inequality follows analogously for displacements $-t_j n$.

Proof. At every restoration step attractivity makes the scalar projection $n \cdot X$ nondecreasing. Telescope the increments. $\square$

The convex-hull theorem is an instance of this corollary. Reflection certificates for triangles and useful pair-bisectors for four points are other instances. A symmetry-forced point is simply the special case $L = \{q\}$.

7 Finite displacement networks and polyhedral chests

Several parallel displacements can be concatenated because one scalar projection remains monotone. Nonparallel displacements are different: after the first move the center is unknown, so its intermediate value cannot be replaced by the original symmetric center. The correct construction retains every intermediate center as an auxiliary variable.

Definition 7.1 (Finite displacement network)**.** A *finite displacement network* consists of

(N1) a finite directed graph whose nodes v are auxiliary configurations P_v;

(N2) a vector variable $x_v \in \mathbb{R}^d$ representing the unknown center at each node;

(N3) for every directed edge $v \to w$, a specification that one input has moved by d_{vw}, producing

$$d_{vw} \cdot (x_w - x_v) \geq 0;$$

(N4) linear equalities or inequalities at selected nodes obtained directly from symmetry, from the convex-hull theorem, or from independently certified node chests;

(N5) one distinguished target node t.

Definition 7.2 (Primitive certificate rules)**.** The *finite symmetry–displacement calculus* used in this paper has the following checkable rules.

(C1) *Symmetry rule.* If a listed Euclidean isometry, together with a listed permutation of the inputs, fixes a node configuration, impose that the node center lies in the isometry's fixed affine subspace.

(C2) *Hull rule.* Impose the facet inequalities supplied by Theorem 4.1. This is a convenient derived macro for repeated applications of the symmetry and parallel-restoration argument.

(C3) *Motion rule.* For an edge on which exactly one input moves by the explicit vector d, impose $d \cdot (x_w - x_v) \geq 0$.

(C4) *Combination rule.* Conjoin finitely many already certified linear systems and eliminate auxiliary center variables by linear projection.

(C5) *Rank rule.* A node chest may be attached only by reference to a completed certificate of strictly smaller rank.

A machine-checkable certificate therefore records the coordinates of every node configuration, the symmetry/permutation data, every moved input and displacement vector, every lower-rank certificate identifier, and the final projection. Each item is finite linear algebra. No assumed existence of a special attractive center and no conclusion being proved may occur as a premise.

To exclude circular reasoning, attached node chests must themselves have a finite well-founded proof under these rules.

Definition 7.3 (Finite certificate)**.** A *finite certificate* is a finite derivation tree or directed acyclic proof graph built from Theorem 7.2. Rank-zero constraints are primitive consequences of the axioms (such as a symmetry-fixed affine subspace or the convex-hull theorem). A rank-$(r+1)$ network may use only node chests already certified at rank at most r. The target projection of the resulting finite linear system is a rank-$(r+1)$ certified chest.

This rank convention is only bookkeeping, but it ensures that a chest can never be justified by a chain that eventually depends on itself.

Theorem 7.4 (Finite network polyhedral theorem)**.** *Every finite certificate produces a polyhedral chest for its target configuration. More precisely, let $\mathcal{P}$ be the feasible polyhedron in the product space of all node-center variables defined by the certificate. Then*

$$C_\Gamma(P_t) = \{x_t : \exists (x_v)_{v \neq t} \text{ with } (x_v)_v \in \mathcal{P}\}$$

contains $X(P_t)$ for every invariant attractive center satisfying the primitive axioms, and $C_\Gamma(P_t)$ is a polyhedron.

Proof. Induct on the proof rank. At rank zero the attached constraints are valid by definition. At the induction step, assign $x_v = X(P_v)$ for any actual attractive center rule. Every edge inequality is exactly the finite attractivity law, while every attached lower-rank chest contains the true node center by the induction hypothesis. Thus the vector of true node-center values is feasible in $\mathcal{P}$, and the true target belongs to its coordinate projection.

The feasible set $\mathcal{P}$ is the intersection of finitely many affine half-spaces and hyperplanes, hence a polyhedron. Linear projections of polyhedra are polyhedra. □

For any support direction n, the linear programme

$$\min \ n \cdot x_t \qquad \text{subject to the complete certificate constraints}$$

produces a valid target half-space. A finite collection of chosen directions therefore produces a certified polyhedral outer approximation. An exact H-representation of the projected polyhedron requires exact projection or facet enumeration, or an independent proof that the complete set of facet normals has been found.

Corollary 7.5 (Redundancy and dual certificates)**.** *Let C be a finite polyhedral chest.*

(i) A proposed inequality $a \cdot x \geq b$ is implied by the current chest constraints if and only if

$$\min_{x \in C} (a \cdot x - b) \geq 0.$$

When testing whether a row already present in a defining system is redundant, that row is first removed and the minimization is performed over the remaining constraints.

(ii) Every valid linear inequality implied by a finite certificate admits a Farkas dual certificate as a nonnegative linear combination of primitive inequalities together with unrestricted linear combinations of equality constraints.

Thus a large family of visually different geometric constraints can be reduced algorithmically to irredundant facets, while the dual variables record a proof of each surviving inequality.

8 The standard chest of a triangle

Let ABC be a nondegenerate planar triangle. Three independently certified regions combine to give a small explicit chest.

8.1 The convex hull

By Theorem 4.1,

$$X(ABC) \in \operatorname{conv}\{A, B, C\}.$$

8.2 The characteristic parallelogram

For side AB, write

$$M_c = \frac{A+B}{2}.$$

Define $\mathcal{S}_c$ as the strip between the perpendicular bisector of AB and the line through C perpendicular to AB. Equivalently,

$$\min\{(B-A)\cdot C, (B-A)\cdot M_c\} \le (B-A)\cdot x \le \max\{(B-A)\cdot C, (B-A)\cdot M_c\}\,. \tag{10}$$

Both boundary half-spaces are reflection certificates. For one boundary, project C orthogonally onto the perpendicular bisector of AB and restore it. For the other, reflect one base vertex across the altitude line through C and restore it. In each predecessor the relevant reflection forces the center onto the reflection axis, and Theorem 6.1 gives the required side of the axis.

Repeat cyclically and put

$$\mathrm{CP}(ABC) = \mathcal{S}_a \cap \mathcal{S}_b \cap \mathcal{S}_c.$$

Proposition 8.1 (Characteristic parallelogram)**.** CP(ABC) *is a chest and is a parallelogram, possibly degenerate.*

Proof. Each strip is a chest, so their intersection is a chest. For the shape statement translate so that $A = 0$, and introduce

$$U = x\cdot B, \qquad V = x\cdot C,$$

which are valid linear coordinates because B, C are independent. Put

$$\alpha = \frac{|B|^2}{2}, \qquad \beta = \frac{|C|^2}{2}, \qquad \gamma = B\cdot C.$$

The first two strips form the rectangle

$$R = \{U \text{ between } \gamma \text{ and } \alpha, \quad V \text{ between } \gamma \text{ and } \beta\}.$$

The third strip is

$$V - U \text{ between } 0 \text{ and } \beta - \alpha. \tag{11}$$

Its two boundary lines pass through the opposite rectangle corners (γ,γ) and (α,β).

The rectangle and the strip (11) are invariant under the half-turn about the rectangle center,

$$(U,V) \longmapsto (\alpha+\gamma-U,\, \beta+\gamma-V),$$

because this map sends $V-U$ to $(\beta-\alpha)-(V-U)$. The other two rectangle corners have third-strip coordinates

$$\beta-\gamma, \qquad \gamma-\alpha,$$

whose sum is $\beta-\alpha$. If γ lies between α and β, both corners satisfy (11); the third strip is redundant and the intersection is the rectangle. Otherwise those two opposite corners are excluded, one by each boundary of the strip, leaving a centrally symmetric quadrilateral. Hence the intersection is again a parallelogram. Degenerate limiting cases are included. The inverse linear coordinate map preserves parallelograms. $\square$

8.3 The internal characteristic Napoleon triangle

Assume first that ABC is positively oriented, and let J denote rotation through $90°$. Put

$$R_{60} = \frac{1}{2}I + \frac{\sqrt{3}}{2}J.$$

Construct the internal equilateral apices

$$\begin{aligned} C_0 &= A + R_{60}(B - A), \\ A_0 &= B + R_{60}(C - B), \\ B_0 &= C + R_{60}(A - C), \end{aligned}$$

so that each apex is on the same side of its base as the opposite vertex. Let

$$N_c = \frac{A + B + C_0}{3}, \qquad N_a = \frac{B + C + A_0}{3}, \qquad N_b = \frac{C + A + B_0}{3}.$$

For a non-equilateral triangle define the three half-planes

$$(C - C_0) \cdot (x - N_c) \geq 0, \tag{12}$$
$$(A - A_0) \cdot (x - N_a) \geq 0, \tag{13}$$
$$(B - B_0) \cdot (x - N_b) \geq 0, \tag{14}$$

and let their intersection be $\mathrm{CNT}(ABC)$. For an equilateral triangle we define $\mathrm{CNT}(ABC) = \{G\}$, where $G = (A+B+C)/3$. This is the continuous geometric extension; the raw displacement inequalities themselves become vacuous when all three displacements are zero, while symmetry directly forces the center to G.

Proposition 8.2 (Internal characteristic Napoleon triangle)**.** $\mathrm{CNT}(ABC)$ *is a chest. For a non-equilateral triangle it is the anticomplementary triangle of the inner Napoleon triangle $N_aN_bN_c$; equivalently, N_a, N_b, N_c are its side midpoints. The inner Napoleon triangle is equilateral. In the equilateral limit,* $\mathrm{CNT}(ABC)$ *degenerates to the centroid.*

Proof. For a non-equilateral triangle, each defining half-plane follows by moving one vertex from the corresponding equilateral predecessor, whose center is forced to N_a, N_b, N_c, respectively. Hence the intersection is a chest.

The geometry can be checked directly. The internal-apex formulas give

$$N_a - N_b = \frac{1}{\sqrt{3}}J(C - C_0), \tag{15}$$

and cyclically. Thus the boundary line through N_c in (12) is parallel to N_aN_b, and similarly for the other two boundaries. Moreover

$$A_0 + B_0 + C_0 = A + B + C,$$

so

$$N_a + N_b + N_c = A + B + C, \tag{16}$$

meaning that the inner Napoleon triangle has the same centroid G as ABC. Since

$$G - N_c = \frac{C - C_0}{3},$$

we have

$$(C - C_0) \cdot (G - N_c) = \frac{1}{3}|C - C_0|^2 > 0, \tag{17}$$

and cyclically. Hence each inequality selects the side of its boundary containing G. The three lines are therefore precisely the sides of the anticomplementary triangle of $N_aN_bN_c$, with the N_i's as side midpoints.

Finally, if $a = |BC|$, $b = |CA|$, $c = |AB|$, and Δ is the area of ABC, then

$$|C - C_0|^2 = \frac{a^2 + b^2 + c^2}{2} - 2\sqrt{3}\,\Delta, \tag{18}$$

and the same expression is obtained cyclically for $|A - A_0|^2$ and $|B - B_0|^2$. By (15) the three sides of $N_aN_bN_c$ therefore have equal length. The standard inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}\,\Delta$, with equality only for an equilateral triangle, shows that this Napoleon triangle is nondegenerate exactly when ABC is non-equilateral. In the equilateral case symmetry forces $X = G$, agreeing with the stated degenerate definition. □

The negatively oriented case follows by reflection equivariance, or by replacing J with $-J$.

Definition 8.3 (Standard triangle chest)**.** The *standard chest* of ABC is

$$\mathcal{C}_{\mathrm{std}}(ABC) = \mathrm{conv}\{A, B, C\} \cap \mathrm{CP}(ABC) \cap \mathrm{CNT}(ABC). \tag{19}$$

Theorem 8.4. *For every nondegenerate triangle, $\mathcal{C}_{\mathrm{std}}(ABC)$ is a chest for the full finite-response class $\mathcal{A}^{\mathrm{fr}}_{3,2}$.*

Proof. Each factor in (19) is a chest. □

8.4 A 13–20–21 example

Take

$$A = (0, 0), \qquad B = (21, 0), \qquad C = (5, 12).$$

The triangle has side lengths $20, 13, 21$ and area 126. The characteristic parallelogram has vertices

$$(10.5, 2.6667),\ (10.5, 4.3750),\ (5, 6.6667),\ (5, 4.9583),$$

while the internal characteristic Napoleon triangle has vertices

$$(5, -0.12436),\ (14.07180, 2.88675),\ (6.92820, 9.23760).$$

Their intersection with ABC is the quadrilateral

$$(10.5, 2.66667), \quad (10.5, 4.37500), \quad (6.28814, 6.12994), \quad (5.96410, 4.55662),$$

of area 7.47204, only 5.9302% of the original triangle.

9 The finite-certificate outer locus and the triangle conjecture

The adjective "optimal" has meaning only relative to a specified proof calculus. The well-founded certificates of Theorem 7.3 provide one such calculus.

Definition 9.1 (Finite certificate library)**.** Let $\mathfrak{L} = \{\Gamma_1, \ldots, \Gamma_m\}$ be a finite list of *completed* well-founded certificates for the same target configuration P. Define

$$C_{\mathfrak{L}}(P) = \bigcap_{j=1}^{m} C_{\Gamma_j}(P).$$

This is a polyhedron by Theorem 7.4. A finite list of reusable generators is not identified with a finite library: closing generators under arbitrarily many compositions can create infinitely many completed certificates, whose total intersection need not be polyhedral.

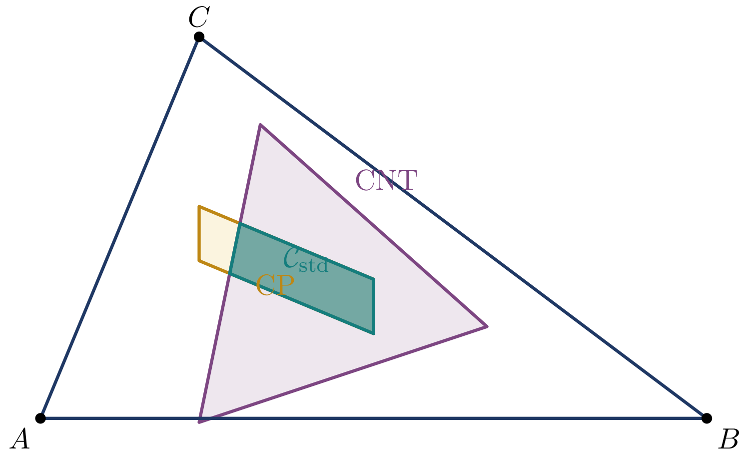

Figure 1: The standard chest for a 13–20–21 triangle.

Definition 9.2 (Full finite-certificate chest)**.** Let $\mathfrak{G}_{\rm fin}$ be the class of all finite well-founded certificates permitted by the symmetry–displacement calculus. Define

$$C_{\rm fin}(P) = \bigcap_{\Gamma \in \mathfrak{G}_{\rm fin}} C_\Gamma(P).$$

Every $C_\Gamma(P)$ is a chest, hence so is their intersection. Unlike a single finite-library chest, $C_{\rm fin}(P)$ need not *a priori* be polyhedral, because it is an infinite intersection of polyhedra.

The exact one-dimensional theorem shows that this calculus can be complete: for real data, the two reflection-symmetric predecessor constructions used in Theorem 2.1 certify $[L, U]$, while explicit continuous attractive means fill the same interval. Thus in one dimension

$$C_{\rm fin}(x) = \mathcal{V}_N(x) = [L(x), U(x)].$$

The following is the corresponding two-dimensional conjecture at arity three.

Conjecture 9.3 (Triangle finite-certificate conjecture)**.** *For every noncollinear planar triangle,*

$$\boxed{C_{\rm fin}(ABC) = \mathcal{C}_{\rm std}(ABC).}$$

The inclusion

$$C_{\rm fin}(ABC) \subseteq \mathcal{C}_{\rm std}(ABC)$$

is immediate because each component of the standard chest has a finite certificate. The content of the conjecture is the reverse inclusion: no finite well-founded symmetry–displacement certificate, however elaborate, cuts the standard chest further.

For a full-dimensional planar configuration and a chosen certified chest $C(P)$, define the confinement ratio

$$\rho_C(P) = \frac{\operatorname{area}(C(P))}{\operatorname{area}(\operatorname{conv} P)}.$$

For the example of Fig. 1,

$$\rho_{C_{\rm fin}} \le 0.059302$$

unconditionally, while Theorem 9.3 would give equality with the standard-chest ratio. We use “confinement ratio” rather than “symmetry score” until a monotone relation with a precise measure of geometric symmetry has been established.

10 Why four points require networks

The transition from three to four points exposes a logical issue that is easy to miss geometrically. Suppose a symmetric anchor Q_0 has known center q_0, and two different vertices must be moved by nonparallel displacements u and v. If the u-move is performed first, the center reaches an unknown intermediate point y. The true inequalities are

$$u \cdot (y - q_0) \geq 0, \qquad v \cdot (x - y) \geq 0.$$

The second inequality is *not* $v \cdot (x - q_0) \geq 0$. Imposing both endpoint inequalities as if both moves started at q_0 can exclude genuine attractive centers.

10.1 Parallelogram strips

For a partition $\{i, j\} \mid \{k, l\}$, put

$$m_1 = \frac{p_i + p_j}{2}, \qquad m_2 = \frac{p_k + p_l}{2}, \qquad d = m_1 - m_2.$$

By translating one pair to a centrally symmetric predecessor and then restoring it by parallel motions, attractivity gives

$$d \cdot m_2 \leq d \cdot x \leq d \cdot m_1, \tag{20}$$

because $d \cdot m_1 - d \cdot m_2 = |m_1 - m_2|^2 \geq 0$. The three unordered $2+2$ partitions therefore produce a basic polyhedral chest after intersection with the convex hull. If $m_1 = m_2$, the configuration is centrally symmetric and every invariant center is that common midpoint.

10.2 Useful pair-bisectors

Let p_i, p_j be a pair with midpoint m and difference $u = p_j - p_i$. If all remaining points lie in one closed half-space bounded by the perpendicular bisector, so that for some $\sigma \in \{-1, 1\}$

$$\sigma\, u \cdot (p_k - m) \geq 0 \qquad (k \neq i, j),$$

project the remaining points onto that bisector. Reflection swaps p_i, p_j and fixes the projected points, forcing the predecessor center to the bisector. Restoring the points uses parallel multiples of σu, hence

$$\sigma\, u \cdot (x - m) \geq 0. \tag{21}$$

These useful-bisector inequalities are independent of the $2+2$ strips in general. For example, for

$$A = (0, 0), \quad B = (4, 7), \quad C = (2, 4), \quad D = (3, 3),$$

the points C, D lie on the B-side of the perpendicular bisector of AB, giving

$$4x + 7y \geq \frac{65}{2}.$$

The basic parallelogram-strip chest contains $(2, 2)$, which violates this inequality. Thus the bisector family genuinely strengthens the $2+2$ family.

10.3 A fully specified four-point computation

Add to the 13–20–21 triangle its centroid:

$$A = (0, 0), \quad B = (21, 0), \quad C = (5, 12), \quad D = \left(\frac{26}{3}, 4\right).$$

The convex hull is still ABC, with area 126. Substitution in (20) gives the three strip pairs

$$-\frac{701}{18} \le \frac{11}{3}x - 8y \le \frac{77}{2}, \tag{22}$$
$$-\frac{3149}{18} \le -\frac{37}{3}x + 4y \le -\frac{41}{6}, \tag{23}$$
$$-\frac{410}{3} \le -\frac{26}{3}x - 4y \le -\frac{410}{9}. \tag{24}$$

Together with the hull inequalities

$$y \ge 0, \qquad 3x + 4y \le 63, \qquad 12x - 5y \ge 0,$$

they define a nine-vertex polygon of area 73.59721831, hence ratio 0.58410491.

All six pairs are useful in the sense of (21). In the pair order AB, AC, AD, BC, BD, CD, their half-planes may be written

$$x \le \frac{21}{2}, \qquad 10x + 24y \ge 169, \qquad 39x + 18y \ge 205,$$
$$4x - 3y \le 34, \qquad 222x - 72y \le 3149, \qquad -66x + 144y \le 701.$$

Three of these cut the strip chest independently. Intersecting all six with (22)–(24) and the hull gives the polygon

$$(2.50715308, 6.01716738), \quad (2.5, 6), \quad (10.5, 2.66666667),$$
$$(10.5, 7.875), \quad (9.00574713, 8.99568966),$$

of area 26.76208315 and ratio 0.21239749.

For the independent two-order calculation, use every permutation (a, b, c, d) of (A, B, C, D) and both equilateral apices

$$c_0 = a + R_{\pm\pi/3}(b - a), \qquad d_0 = \frac{a + b + c_0}{3}.$$

Put $u = c - c_0$, $v = d - d_0$. For the order $c_0 \to c$ followed by $d_0 \to d$, retain the intermediate center z and impose

$$u \cdot (z - d_0) \ge 0, \qquad v \cdot (x - z) \ge 0,$$

together with the parallelogram-strip chest of (a, b, c, d_0). Reverse the motion order and repeat. Eliminating z by linear programming over the 48 permutation–orientation scenarios gives 45 distinct normalized support half-planes. Their intersection with the elementary strip chest is the six-vertex polygon

$$(5.09682665, 0.34576449), \quad (5.25641026, 0), \quad (5.37465597, 0),$$
$$(14.07179677, 2.88675135), \quad (7.81447067, 8.44968795),$$
$$(6.65662313, 7.91900782),$$

with area 38.80897657 and ratio 0.30800775. Finally, intersecting this network chest with the useful-bisector chest gives

$$(5.96410162, 4.55662433), \quad (10.5, 2.66666667),$$
$$(10.5, 6.06217783), \quad (7.81447067, 8.44968795),$$
$$(6.65662313, 7.91900782),$$

whose area is 16.90001251 and whose hull-area ratio is

$$\boxed{0.13412708.}$$

The vertices are listed counterclockwise. The supplementary `verify_four_point.py` reconstructs the full calculation in exact arithmetic in $\mathbb{Q}(\sqrt{3})$, without a numerical geometry library. It enumerates the intermediate polygons, minimizes each linear functional at their vertices, and records the originating scenario and active inequalities. The included JSON certificate and verification transcript give exact region vertices, areas, all derived unnormalized inequalities, and successful checks. The original Wolfram implementation and numerical driver are retained as a separate numerical reproduction route; their tolerance decisions are not used by the exact verifier. The reported decimals are rounded displays of the exact calculation. Run instructions and file hashes are included in the supplement.

Candidate	Status	Area ratio	Certificate or additional assumption
Parallelogram strips	rigorous	58.4105%	Hull and three $2+2$ parallel-motion strips.
Useful bisectors	rigorous	21.2397%	Six reflection-bisector half-planes; three independently cut the strip chest.
Two-order network	rigorous	30.8008%	Intermediate variables, lower-rank strip chests, 48 scenarios and linear projection.
Bisector $\cap$ network	rigorous	13.4127%	Intersection of two independently valid certificate families.
Literal union–OR	unproved	30.8008% here	Assumes that two nonparallel moves imply a Boolean union of endpoint half-planes.
Direct "both" intersection	false	0.0663%	Treats both moves as if they began at the anchor; it excludes the centroid.
Recursive absorption	conditional	58.4105% here	Adds the variable-arity absorption law $X_4(A,B,C,X_3(A,B,C)) = X_3(A,B,C)$.

Table 1: Logical status and numerical size of seven four-point candidates. Equal numerical areas do not imply equivalent proofs.

The direct-intersection failure is decisive. The ordinary four-point centroid is $(26/3, 4)$ and has

$$J_i = \frac{1}{4} I_2 \succeq 0.$$

It is attractive, yet the direct "both" region excludes it. Any proposed chest with that property is false. The literal union–OR construction is not disproved by this test, but it is not a consequence of separate attractivity: after the first nonparallel motion, the second inequality is anchored at an unknown intermediate center. The recursive construction is mathematically natural for *stable* means, but absorption is an additional axiom rather than a theorem of attractivity. These distinctions explain why intermediate center variables are part of the logic, not merely a computational nuisance.

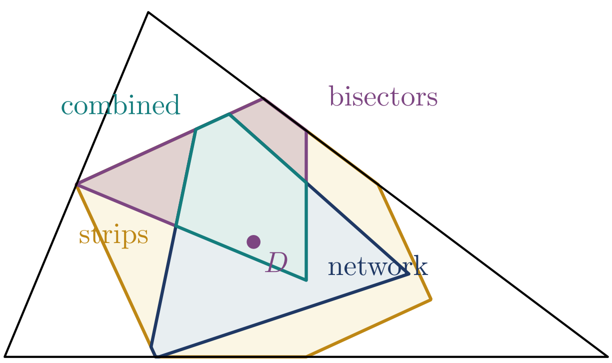

Figure 2: The reproducible four-point comparison of Section 10.3. Gold is the elementary strip chest, plum the useful bisector chest, blue the two-order network chest, and teal their rigorous intersection.

11 Smooth hearts and the triangle-center laboratory

The outer theory above never required smoothness. The heart problem is different: a point of a smooth heart must be realized by an entire globally admissible center function.

Proposition 11.1 (Barycentric form for triangle centers)**.** *For a noncollinear triangle ABC, every invariant center rule has a unique affine representation*

$$X(A,B,C) = f_A A + f_B B + f_C C, \qquad f_A + f_B + f_C = 1.$$

The coefficients are dimensionless functions of the congruence class of the triangle. Permutation invariance makes them permutation-covariant; with a chosen orientation-free convention they may therefore be expressed in terms of the three side lengths. If the center rule is C^r or analytic on triangle-shape space, the normalized coefficient functions have the same regularity away from singular strata.

Proof. Because A, B, C are affinely independent in their plane, every point has unique barycentric coordinates with coefficient sum one. Translation invariance removes dependence on the origin, orthogonal equivariance removes orientation and coordinate-frame dependence, and scale equivariance makes the coefficients dimensionless. The only remaining Euclidean similarity data of an unordered triangle are its side lengths up to common scale, while permutation invariance induces the corresponding covariance of the coefficients. □

Kimberling's center-function formalism and the Encyclopedia of Triangle Centers provide a large explicit catalogue of such invariant functions [1, 2].

A point y belongs to the C^1 heart of a fixed triangle only if there is a *global* center rule $X \in \mathcal{A}^1_{3,2}$ with $X(ABC) = y$. A feasible value and Jacobian at one triangle are not sufficient.

The Jacobian criterion gives the global differential condition

$$\mathrm{Sym}(D_A X) \succeq 0, \qquad \mathrm{Sym}(D_B X) \succeq 0, \qquad \mathrm{Sym}(D_C X) \succeq 0$$

throughout triangle-shape space. By permutation symmetry it is enough, in a global proof over all labelled triangles, to establish one vertex condition and obtain the others by relabelling. At one fixed labelled triangle, however, positivity of two blocks does not follow algebraically for the third merely from $J_A + J_B + J_C = I$.

Constructed parametric families whose symmetric Jacobians are globally positive semidefinite provide certified heart points. Conversely, a finite shape grid or a local semidefinite jet computation supplies only evidence or an outer relaxation until global integrability and positivity are proved.

For a fixed regularity class $\mathcal{A}^r$, the support function of the closed attainable locus is

$$h_P^{(r)}(n) = \sup_{X \in \mathcal{A}^r} n \cdot X(P), \qquad \|n\| = 1.$$

This suggests an inner–outer programme:

(1) construct globally attractive center families to raise certified inner support values;

(2) strengthen chest or local-relaxation bounds to lower certified outer support values;

(3) when the two agree in a direction, identify an exact supporting value and contact face; exposure of a point additionally requires a singleton face.

A central regularity question is whether, for triangles,

$$\overline{\mathcal{V}}_{\mathcal{A}^\omega}(P) \stackrel{?}{=} \overline{\mathcal{V}}_{\mathcal{A}^\infty}(P) \stackrel{?}{=} \overline{\mathcal{V}}_{\mathcal{A}^1}(P).$$

12 Relation to aggregation theory and triangle geometry

12.1 Exact relation to SC-monotonicity

The response law (1) is not new under a different name. Pérez-Fernández, De Baets, and Gagolewski study general aggregators $A : (\mathbb{R}^d)^N \to \mathbb{R}^d$ and separate two choices: move one input or all inputs, and require the output either to move parallel to the input direction or merely not to oppose it [6]. With $t > 0$, a direction u, and ΔA denoting the output increment, their four notions are:

Name	Inputs moved	Required output response
SP	one point by tu	$\Delta A = ku$ for some $k \geq 0$
MP	all points by the same tu	$\Delta A = ku$ for some $k \geq 0$
SC	one point by tu	$u \cdot \Delta A \geq 0$
MC	all points by the same tu	$u \cdot \Delta A \geq 0$

Thus attractivity is *exactly SC-monotonicity*: set $h = tu$ in (1). SP implies both MP and SC, and either MP or SC implies MC. In one dimension SP and SC reduce to ordinary monotonicity. For a translation-equivariant center, MP holds automatically because translating all inputs by tu translates the output by exactly tu; SC is the substantive single-input condition.

Their examples are directly relevant. The centroid satisfies all four properties. The spatial median and the Euclidean minimum-enclosing-ball center satisfy common-motion properties but fail SC; their orthomedian is SC-monotone. The coordinatewise median is SC-monotone but is not equivariant under arbitrary rotations. This last example shows why the present paper keeps monotone response and Euclidean invariance as separate axioms: SC alone does not make an aggregate geometrically intrinsic.

There is also a useful contrast concerning internality. Their Proposition 39 derives convex-hull internality from ultramonotonicity plus idempotence. The facet-projection argument of Theorem 4.1 derives it instead from the weaker SC condition together with Euclidean equivariance. Reflection symmetry of the projected configuration supplies the boundary condition that is unavailable for a completely general aggregator.

The difference in objectives is equally important. Pérez-Fernández et al. provide a logical taxonomy and classify aggregation functions. They do not impose permutation symmetry, homogeneity, or the triangle-center framework, and they do not ask for the locus of *all* admissible outputs at one fixed configuration. The chest/heart distinction, the finite-certificate OSC, the characteristic triangle regions, and the symmetry–displacement network calculus are therefore additional geometric questions, not renamed results from their paper. A freely accessible accepted manuscript is linked in the bibliography alongside the DOI.

12.2 Adjacent literatures and the present contribution

Penalty-based multidimensional aggregation already places vector summaries in a broad framework that includes many classical location rules [3]. Multivariate location theory separately emphasizes robustness and transformation equivariance; affine-equivariant medians form a substantial literature [4]. The present paper uses those traditions as context but asks a different inverse question: after fixing the data, what geometry is forced simultaneously on every invariant SC-monotone rule?

The present programme adds a configuration-dependent geometric layer. We fix the input set and ask for

- the necessary locus forced by invariance and SC-monotonicity;
- the attainable locus realized by globally admissible center maps;

- finite proof systems that generate explicit outer inequalities;
- the geometry and limiting behaviour of these loci as the configuration varies or becomes dense.

For three planar points the problem meets classical triangle geometry. Kimberling's center-function framework supplies explicit invariant functions, and the ETC supplies a uniquely large experimental source of candidate centers [1, 2]. The characteristic parallelogram and the internal characteristic Napoleon triangle arise here not as named-center constructions but as regions forced simultaneously on *all* attractive invariant centers.

Replacement and absorption principles for means are logically distinct from attractivity. Stability and barycentric associativity [7, 5] may be imposed on variable-arity families, but they are additional axioms. In particular, inserting a three-point center as a fourth datum does not, from attractivity alone, force the four-point center to remain fixed.

13 Distributional stability and the region-level programme

The long-term motivation is not limited to finite configurations. One would like a coherent notion of centrality for a geometric body obtained as the limit of many points filling that body at approximately constant density. The one-dimensional result Theorem 2.3 shows that this programme can be made exact in the simplest case.

Let $R \subset \mathbb{R}^d$ be a bounded region of positive volume and write

$$\mu_R = \frac{\mathbf{1}_R(x)\,\mathrm{d}x}{\mathrm{Vol}(R)}$$

for normalized constant-density measure. For a point cloud $P_N = (p_1, \ldots, p_N) \subset R$, let

$$\mu_{P_N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{p_i}$$

be its empirical measure.

Weak convergence alone is too permissive for an outer centrality theory: a vanishing fraction of geometrically exceptional points can have negligible mass while still changing a convex hull or a finite chest. A useful stronger notion is the L^∞ transport distance

$$W_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \operatorname*{ess\,sup}_{(x,y)\sim\pi} \|x - y\|.$$

We therefore make the following provisional definition.

Definition 13.1 (Uniform filling and stability)**.** A sequence of equal-weight point clouds $P_N \subset R$ *fills R uniformly* if

$$W_\infty(\mu_{P_N}, \mu_R) \longrightarrow 0.$$

Let $K_N(P_N)$ be a chosen compact centrality locus at arity N, for example the closed attainable locus of a specified attractive class or a canonically selected chest. The theory is *stable on R* if there is a compact set $K(R)$, depending only on R and the chosen axioms, such that every uniformly filling sequence satisfies

$$d_H(K_N(P_N), K(R)) \longrightarrow 0.$$

The set $K(R)$ is then the limiting centrality region of R.

For $R = [a, b]$ and $K_N = \mathcal{V}_N$, Theorem 2.3 proves

$$\boxed{K([a,b]) = \left\{\frac{a+b}{2}\right\}.}$$

Thus constant-density filling collapses the entire one-dimensional heart/chest, not merely one selected mean, to the symmetry center.

The higher-dimensional problem is subtler. A disk or centrally symmetric body suggests a point limit by symmetry, while a generic region may retain a nontrivial limiting heart or chest. The appropriate mode of convergence may also depend on which outer or inner theory is being studied. Definition 13.1 should therefore be regarded as the first precise target, not as the final word.

13.1 Coherent families and three meanings of stability

The input object must be specified. A finite multiset, its unweighted support, and its empirical probability measure are different objects. Repeating inputs changes empirical masses while leaving the support unchanged. The body theory below uses probability measures; normalized volume describes uniform material. For variable-arity rules a natural compatibility requirement is replication invariance, $X_{mN}(P, \ldots, P) = X_N(P)$. For weighted inputs, splitting an atom into coincident atoms with the same total mass must likewise leave the value unchanged. These are additional requirements, not consequences of fixed-arity attractivity. Neither requirement asserts barycentric associativity or absorption.

Perturbation continuity, resistance to small amounts of contamination, and convergence of an entire attainable region are distinct properties. In particular, continuity of every member of a class separately does not establish uniform control over the class. The following sufficient framework makes the quantifiers explicit.

Proposition 13.2 (Uniform stability of a common family)**.** *Let $\mathfrak{A}$ be a nonempty common family of single-valued rules on a metric space of probability measures $(\mathcal{M}, d)$. Suppose*

$$\|T(\mu) - T(\nu)\| \le \omega(d(\mu, \nu)) \quad (T \in \mathfrak{A}), \qquad \omega(t) \longrightarrow 0 \quad (t \downarrow 0).$$

If $K(\mu) = \overline{\{T(\mu) : T \in \mathfrak{A}\}}$ is compact for every μ, then

$$d_H(K(\mu), K(\nu)) \le \omega(d(\mu, \nu)).$$

If the family is closed under constant convex mixtures, these loci are convex.

Proof. Pair $T(\mu)$ with $T(\nu)$ for the same rule T. The assumed bound controls the distance from every attained value to the other locus. Passing to closures and reversing μ, ν proves the Hausdorff bound. Convexity follows from mixing. □

This is a sufficient axiom for a selected stable class, not a theorem that the unrestricted finite-response class automatically satisfies it. One must also identify the finite-cloud rules with restrictions of the common measure rules; otherwise the class itself may change with arity.

Example 13.3 (Failure of weak continuity for unrestricted loci)**.** For $N \ge 3$, take $N - 1$ zeros and one input at 1. The exact one-dimensional locus is $[0, 1/2]$, by Theorem 2.1, although the empirical measures converge weakly to δ_0, whose center is necessarily 0. This remains a counterexample on the fixed compact interval $[0, 1]$. The W_∞ distance to δ_0 is 1, so the example does not contradict the stronger filling topology. It demonstrates why the topology is a substantive part of centrality.

Proposition 13.4 (Translation obstruction for unbounded sets)**.** *There is no assignment of a nonempty compact region to every nonempty subset of* $\mathbb{R}^d$ *that is equivariant under all translations.*

Proof. The set $\mathbb{Z}^d$ is unchanged by a unit lattice translation. Its assigned region would therefore satisfy $K = K + e_1$, and contain $x + ne_1$ for all integers n whenever it contains x. This contradicts compactness. □

Thus increasing arity in a bounded body must be distinguished from arbitrary unbounded inputs. For unbounded probability measures one needs tail or moment assumptions; for unweighted infinite sets an exhaustion procedure must be specified and its dependence studied. Some inputs have no finite central region. For centrally symmetric measures, equivariance forces every single-valued rule to the symmetry center. Under a uniform family-continuity hypothesis this also proves collapse of approximately symmetric clouds. Paper III supplies a concrete region-level convergence theorem for the convex family generated by admissible power centers on compactly supported measures.

14 Physical interpretation and broader programme

The adjective "attractive" is a geometric response metaphor, not a claim that every admissible center arises from a physical force law. Nevertheless, with all other inputs fixed, the map

$$p_i \longmapsto X(p_1, \ldots, p_i, \ldots, p_N)$$

has nonnegative incremental longitudinal response: displacing one source by h cannot produce an output change with negative projection along h. This resembles a positive static susceptibility or incremental response law.

Centrality throughout is relative to the specified axioms; excluded rules may still be useful centers for other objectives. Given finite points in $\mathbb{R}^d$, invariance and positive response exclude parts of space from being central; a chest records those exclusions. A heart records locations actually realizable by global response laws of a chosen regularity. As the number of points increases, stability asks whether these regions themselves converge to a geometric centrality region of the body being filled.

The triangle is therefore not the endpoint of the theory. It is the smallest planar laboratory in which the chest is genuinely two-dimensional, classical center functions are abundant, and exact constructions remain visible between the one-dimensional theory of means and the eventual theory of continuous geometric bodies.

15 Open problems

The first stage of the programme leads to the following concrete questions.

1. Prove or disprove Theorem 9.3: does every finite well-founded symmetry–displacement certificate for a triangle become redundant after the convex hull, characteristic parallelogram, and internal characteristic Napoleon triangle are imposed?

2. Give an intrinsic description of the facets of finite network chests and classify which symmetric anchors generate genuinely new irredundant inequalities.

3. Determine whether the full finite-certificate chest $C_{\mathrm{fin}}(P)$ is polyhedral for triangles and for four planar points despite being defined as an infinite intersection of finite polyhedral chests.

4. Determine the exact relation between $C_{\mathrm{fin}}(P)$ and the weak finite-response attainable locus $\mathcal{V}_{\mathcal{A}^{\mathrm{fr}}}(P)$. Is the finite certificate calculus complete for all linear inequalities, as it is in one dimension?

5. Determine whether the closures of the analytic, smooth, and C^1 triangle hearts coincide.

6. Construct globally certified smooth or analytic triangle-center families whose support values meet proved chest bounds, thereby identifying exact exposed points or boundary arcs of the triangle attainable locus.

7. Develop the four-point calculus systematically and quantify how much useful pair-bisectors, centrally symmetric anchors, equilateral-plus-center anchors, coincident-point strata, and longer motion networks improve the basic convex-hull chest.

8. Determine which finite-point centrality loci are stable in the sense of Theorem 13.1. For a uniformly filled convex body R, does the full finite-response attainable locus have a Hausdorff limit independent of the chosen filling sequence?

9. For bodies without central symmetry, determine whether the limiting heart/chest is generally a nontrivial region or collapses to a distinguished point. Relate the answer to geometric and physical notions of response and symmetry.

16 Conclusion

The theory begins with an exact fact about ordinary real data. For N reals, invariant monotonic response confines every center to the interval between the extreme symmetric order-statistic averages, and explicit continuous attractive means fill that interval completely. For three numbers this is the interval between the median and the midrange. Under uniform filling of an interval, the whole interval of admissible centers converges to the midpoint.

The same proof mechanism survives the passage from numbers to vectors. Euclidean equivariance and attractivity force convex-hull internality; symmetry-fixed affine subspaces and vertex restorations produce linear support inequalities; and finite well-founded networks of nonparallel moves produce polyhedral chests after elimination of intermediate center variables.

The four-point example makes that proof calculus auditable. Starting with a hull-area ratio of 58.4105% for the elementary strips, an independent reflection-bisector family gives 21.2397%, and its intersection with the two-order network gives 13.4127%. Explicit inequalities, polygon vertices, logical status labels, and a companion reproduction script are provided. The same example cleanly separates rigorous certificates from the unproved union–OR shortcut, the false direct-intersection rule, and the additional absorption axiom.

For triangles, the convex hull, characteristic parallelogram, and internal characteristic Napoleon triangle give a small explicit standard chest. We conjecture that the finite certificate calculus is already complete for this outer problem, just as it is in the one-dimensional prototype. The complementary inner problem is to determine how much of that region is filled by globally smooth or analytic attractive triangle centers.

The long-term object is a geometry of centrality that persists as finite point clouds become dense. Chests express what invariance and positive response force; hearts express what coherent center laws can realize; stability asks whether those regions converge when points fill a geometric body at constant density. In that sense the classical theory of means, triangle centers, polyhedral certificates, and physical response are different scales of one problem.

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