Attractive Centers II

Isosceles Geometry, Local Admissibility, and Global Extension

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Abstract

Paper I developed the outer geometry of attractive centers through symmetry and finite displacement inequalities. This paper studies the complementary extension problem for smooth and analytic triangle centers, using the isosceles stratum as an exact symmetry-reduced laboratory.

The isosceles symmetry-reduced ambient certificate chest is the interval joining the geometric median M_1 and the minimum-enclosing-ball center M_∞; the power-distance family sweeps this interval when motion is constrained to the symmetry stratum. Locally, every prescribed axial value admits a strictly PSD-admissible analytic invariant germ, whereas rotational lifting of one-dimensional regular means gives globally attractive centers but fills only a strict subinterval. In complex shape coordinates $A = 0, B = 1, C = z$, ambient attractivity is equivalent to three scalar first-order cone inequalities for one function $f(z, \bar{z})$, and holomorphic rigidity forces every globally attractive, permutation-invariant holomorphic center with the standing extension property to be the centroid. The isosceles chest endpoints are not attained by analytic attractive centers at non-equilateral shapes; their membership in the closure remains conjectural. The axial chest and constrained power-sweep theorem extend to every regular-base axial simplex. Sharp normalized response intervals at the flat and tall ends of the isosceles ray provide inexpensive necessary tests for explicit center formulas. A corrected exploratory ETC scan illustrates these tests but is not used to prove attractivity. The results isolate the remaining question as a global analytic continuation problem rather than a pointwise semidefinite one.

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1 Introduction

The project began with a question about means. On the real line, a regular mean is constrained by monotonicity and invariance; for three inputs its exact invariant attractive locus is the interval between the median and the midrange. The original route from this observation to triangle geometry was to regard complex numbers as vectors in $\mathbb{R}^2$: an invariant average of three complex numbers is then naturally a triangle center. This suggested replacing one-dimensional monotonicity by a Euclidean response condition and asking for "regular means" of three vectors.

Paper I [1] developed the corresponding outer problem for vector-valued means: if one input point is displaced by h, the center is required to satisfy

$$h \cdot \big[X(P_{i,h}) - X(P)\big] \geq 0. \tag{1}$$

For three planar inputs, this brings the subject into direct contact with classical triangle centers and their barycentric conventions [4, 5]. The displacement law is the single-point consistent monotonicity (SC-monotonicity) of multidimensional aggregation theory [6]; the new question pursued here is the configuration-dependent geometry of all values attained by invariant smooth SC-monotone maps.

The outer theory does not require smoothness. Symmetry, Euclidean equivariance, and (1) produce linear inequalities and hence certified chests. The inner problem is fundamentally different. To prove that one point of a fixed triangle lies in a smooth heart, it is not enough to choose a plausible value and a positive local Jacobian. One must construct a single center function, defined on the entire nondegenerate triangle domain, that is permutation-equivariant, similarity-equivariant, and attractive for every possible vertex displacement.

Standing domain convention: rules are continuous on all configurations, including collisions and collinear inputs, and have the stated regularity on the nondegenerate triangle domain. Finite attractivity applies across degeneracies by this extension; regularity at the total collision is not required. We therefore fix the hierarchy

$$\mathcal{A}^\omega \subseteq \mathcal{A}^\infty \subseteq \mathcal{A}^1,$$

where the superscripts indicate analytic, smooth, and C^1 attractive triangle centers. For a fixed triangle $P = ABC$ we write

$$\mathcal{V}^r(P) = \{X(P) : X \in \mathcal{A}^r\}, \qquad r \in \{1, \infty, \omega\}.$$

Each class is closed under convex mixing, so each attainable locus is convex. The main object of this paper is the gap between the symmetry-reduced chest and the values attained by global analytic center maps.

The paper is organized around one question: which locally admissible values extend to globally attractive triangle-center maps? We first solve the symmetry-reduced isosceles geometry and contrast constrained power sweeps, locally PSD-admissible germs, and globally attractive rotational lifts. The same extension problem is then reformulated by exact complex cone inequalities. The companion Paper IV develops centroidal-ray families, near-equilateral jets, polynomial covariants, and semidefinite inner–outer approximation for arbitrary triangles.

status used below	meaning
Global theorem	One invariant center map is defined on the full nondegenerate triangle domain and satisfies the global attractivity requirement.
Constrained theorem	The rule is tested only under motions preserving a stated symmetry stratum.
Local PSD-admissible germ	An invariant/equivariant map is defined only on a neighborhood in shape space and has PSD symmetric vertex Jacobians there; it is *not* an element of $\mathcal{A}^r$ unless it extends globally.
Numerical observation	A finite computation used for exploration only; it is not sufficient evidence for a global theorem.
Conjecture	A global statement not proved in this paper.

The principal notation is

$$\mathcal{A}^r = \{\text{global attractive invariant centers of class } C^r\}, \qquad \mathcal{V}^r(P) = \{X(P) : X \in \mathcal{A}^r\},$$

with $r \in \{1, \infty, \omega\}$. A *chest* is a proved outer necessary locus; a *certified heart* is a proved convex subset of an attainable locus; $\overline{\mathcal{V}^r(P)}$ is the corresponding closed attainable locus. Orbit-attractivity and local PSD-admissibility are always identified explicitly and are never used as synonyms for global attractivity.

2 Smooth triangle centers on shape space

2.1 Barycentric representation

Let $A, B, C \in \mathbb{R}^2$ be noncollinear and put

$$a = |B - C|, \qquad b = |C - A|, \qquad c = |A - B|.$$

Every point of the affine plane has unique barycentric coordinates with respect to ABC. We follow the standard triangle-center barycentric conventions of Kimberling [4].

Proposition 2.1 (Invariant barycentric form)**.** *Let X be a continuous Euclidean- and similarity-equivariant center of nondegenerate triangles. Then*

$$X(A, B, C) = w_A(a, b, c)A + w_B(a, b, c)B + w_C(a, b, c)C, \qquad w_A + w_B + w_C = 1, \tag{2}$$

where the weights are dimensionless. Permutation invariance is equivalent to permutation covariance of the weight vector. If X is C^r or analytic, so are the weights on the side-length shape domain.

Proof. The barycentric coordinates exist and are unique because A, B, C are affine independent. Translation and rotation leave them unchanged. Positive scaling changes all three side lengths by the same factor while leaving the weights unchanged, so they are homogeneous of degree zero. Uniqueness of barycentric coordinates transfers every vertex permutation to the same permutation of the weights. Regularity follows from solving the nonsingular affine coordinate system locally. □

For a globally attractive center, Paper I's convex-hull theorem implies

$$w_A, w_B, w_C \geq 0. \tag{3}$$

Thus nonnegative barycentric weights are a consequence of attractivity rather than an independent axiom.

2.2 Two useful shape coordinates

Two coordinate systems will be used for different purposes.

For local analysis near the equilateral triangle, define logarithmic shape coordinates

$$\ell_a = \log a, \quad \ell_b = \log b, \quad \ell_c = \log c, \qquad u_i = \ell_i - \frac{\ell_a + \ell_b + \ell_c}{3}. \tag{4}$$

Then

$$u_a + u_b + u_c = 0,$$

and similarity scaling disappears exactly.

For global polynomial constructions, put

$$Q = a^2 + b^2 + c^2, \qquad s_1 = \frac{a^2}{Q}, \quad s_2 = \frac{b^2}{Q}, \quad s_3 = \frac{c^2}{Q}. \tag{5}$$

Let

$$e_2 = s_1 s_2 + s_2 s_3 + s_3 s_1, \qquad e_3 = s_1 s_2 s_3.$$

Heron's identity gives

$$\frac{16\Delta^2}{Q^2} = 4e_2 - 1, \tag{6}$$

where Δ is the triangle area. Consequently the labelled nondegenerate shape domain is exactly

$$\mathscr{S} = \left\{ s_i > 0 : \ s_1 + s_2 + s_3 = 1, \quad e_2 > \frac{1}{4} \right\}. \tag{7}$$

The equilateral triangle is $s_1 = s_2 = s_3 = 1/3$, while $e_2 = 1/4$ is the ordinary degenerate boundary. The closure of (7) inside the unit simplex is compact.

2.3 The Jacobian formula

Write $w = (w_A, w_B, w_C)$ as a function of (a, b, c). Differentiating (2) with respect to A gives

$$J_A = w_A I_2 + (B - A)(\nabla_A w_B)^T + (C - A)(\nabla_A w_C)^T, \tag{8}$$

where differentiated normalization was used to eliminate the A-term. Since a is independent of A,

$$\nabla_A w_j = \frac{\partial w_j}{\partial b} \frac{A - C}{b} + \frac{\partial w_j}{\partial c} \frac{A - B}{c}. \tag{9}$$

The two cyclic formulas are analogous.

Proposition 2.2 (Smooth attractivity criterion)**.** *A C^1 invariant triangle center is attractive on the nondegenerate domain if and only if*

$$S_A = \operatorname{Sym} J_A \succeq 0, \qquad S_B = \operatorname{Sym} J_B \succeq 0, \qquad S_C = \operatorname{Sym} J_C \succeq 0 \tag{10}$$

throughout that domain, with continuous extension used when a finite vertex motion passes through an excluded degeneracy.

This is the Jacobian criterion of Paper I. Permutation covariance means that it is enough in principle to prove one vertex inequality for every *labelled* triangle: the other two follow by relabelling. For numerical work it is often preferable to calculate all three as an internal check.

A convenient canonical chart fixes

$$A = (0, 0), \qquad B = (1, 0), \qquad C = (x, y), \qquad y > 0. \tag{11}$$

Every oriented nondegenerate triangle is directly similar to one and only one point of this chart after choosing the ordered side AB.

3 The smooth heart and its support function

Definition 3.1 (Regularity-dependent heart)**.** For $r \in \{1, \infty, \omega\}$, the C^r attainable locus at P is

$$\mathcal{V}^r(P) = \{X(P) : X \in \mathcal{A}^r\}.$$

Because $\mathcal{A}^r$ is closed under convex mixing, $\mathcal{V}^r(P)$ is convex. We call any explicitly certified convex subset a C^r *heart*. Its closure is denoted $\overline{\mathcal{V}^r(P)}$.

The closures satisfy

$$\overline{\mathcal{V}^\omega(P)} \subseteq \overline{\mathcal{V}^\infty(P)} \subseteq \overline{\mathcal{V}^1(P)}.$$

Whether these three closed loci agree is one of the basic open questions.

A closed convex locus is determined by its support function

$$h_P^r(n) = \sup_{X \in \mathcal{A}^r} n \cdot X(P), \qquad \|n\| = 1. \tag{12}$$

For a C^2 strictly convex boundary in the plane, with $n(\theta) = (\cos\theta, \sin\theta)$,

$$x(\theta) = h(\theta)n(\theta) + h'(\theta)(-\sin\theta, \cos\theta). \tag{13}$$

Thus inner and outer support estimates are a natural common currency: when a globally certified center attains an outer support bound, an exact exposed point of the closed heart has been proved.

Paper I supplies an outer region

$$\overline{\mathcal{V}^r(ABC)} \subseteq \mathcal{C}_{\mathrm{std}}(ABC),$$

where $\mathcal{C}_{\mathrm{std}}$ is the intersection of the triangle, the characteristic parallelogram, and the internal characteristic Napoleon triangle. The present paper will add stronger smooth outer relaxations and, more importantly, certified inner subsets.

4 The isosceles stratum and higher-dimensional kites

The symmetry stratum of isosceles triangles is special for two reasons. First, reflection symmetry reduces every invariant center value to one scalar coordinate. Second, the Euclidean power means, which are not globally attractive for all exponents, become monotone for *every* $p > 1$ when the deformation is constrained to preserve the symmetry. Both facts extend without essential change to every Euclidean dimension.

4.1 Regular-base axial simplices

Fix $d \geq 2$, a unit vector $e \in \mathbb{R}^d$, and vectors $v_1, \dots, v_d \in e^\perp$ forming a regular $(d-1)$-simplex of circumradius $R > 0$:

$$\sum_{j=1}^{d} v_j = 0, \qquad \|v_j\| = R, \qquad v_i \cdot v_j = -\frac{R^2}{d-1} \quad (i \neq j). \tag{14}$$

For $h > 0$ define

$$K_d(R, h) = \{A, v_1, \dots, v_d\}, \qquad A = he. \tag{15}$$

We call this a *regular-base axial simplex*; geometrically it is the natural d-dimensional analogue of an isosceles triangle. The base symmetry group fixes exactly the axis $\mathbb{R}e$, so every invariant center has the form

$$X(K_d(R, h)) = y\, e.$$

The member of this family which is a regular d-simplex has

$$h_* = R\sqrt{\frac{d+1}{d-1}}, \qquad g_* = \frac{h_*}{d+1} = \frac{R}{\sqrt{d^2-1}}. \tag{16}$$

Here $g_* e$ is its centroid, and therefore the uniquely symmetry-forced value of every invariant center.

4.2 Constrained versus ambient attractivity

The word "attractive" has two different meanings on a symmetry stratum and they must not be confused.

Definition 4.1 (Orbit-attractivity on the kite family)**.** A center rule restricted to the family (15) is *orbit-attractive* if it has nonnegative longitudinal response under the elementary symmetry-preserving motions:

(O1) motion of the apex parallel to e with the regular base fixed;

(O2) common translation of the entire base orbit parallel to e, with the apex fixed;

(O3) radial change of the regular base inside $e^\perp$.

For a C^1 axial rule $X = y(R,h)e$, the first two conditions are equivalent to

$$0 \le \partial_h y(R,h) \le 1. \tag{17}$$

The radial base motion is orthogonal to the axial center displacement and therefore has zero longitudinal work.

Ambient attractivity is stronger: it permits an individual vertex to move in an arbitrary direction, generally leaving the kite family. A rule can be orbit-attractive on every kite without being an attractive center on the full configuration space. This distinction is exactly what allows the complete power family below.

4.3 The symmetry-reduced ambient chest

Two natural variational centers already determine the relevant interval. The geometric median belongs to the classical Fermat–Weber/Fréchet location tradition, while M_∞ is the Chebyshev or minimum-enclosing-ball center [7, 8, 9].

Proposition 4.2 (The two variational endpoints)**.** *For $K_d(R,h)$, the geometric median M_1 and the minimum-enclosing-ball center M_∞ lie on the axis at heights*

$$y_1(R,h) = \min\left\{h, \frac{R}{\sqrt{d^2-1}}\right\}, \tag{18}$$

$$y_\infty(R,h) = \max\left\{0, \frac{h^2-R^2}{2h}\right\}. \tag{19}$$

They coincide at the regular-simplex height $h = h_*$.

Proof. By symmetry both minimizers lie on the axis. For the geometric median, minimize

$$\phi_1(y) = h - y + d\sqrt{R^2+y^2}, \qquad 0 \le y \le h.$$

Its interior stationary point satisfies

$$-1 + \frac{dy}{\sqrt{R^2+y^2}} = 0,$$

hence $y = R/\sqrt{d^2-1} = g_*$; if the apex lies below that value the minimum is attained at the apex itself. This proves (18).

For the minimum enclosing ball, symmetry again reduces the radius to

$$\max\{h-y, \sqrt{R^2+y^2}\}.$$

If $h \le R$, the regular base already forces radius at least R, attained at $y=0$. If $h > R$, the optimum equalizes the apex and base distances:

$$h - y = \sqrt{R^2+y^2},$$

which gives (19). At $h = h_*$, both formulas equal g_*. □

We now derive the same interval from the ambient symmetry–displacement calculus of Paper I.

Theorem 4.3 (Axial kite chest)**.** *Let X be any globally invariant attractive center defined on all configurations needed by the displacement certificates. At the target $K_d(R,h)$,*

$$\boxed{X(K_d(R,h)) \in \operatorname{conv}\{M_1(K_d(R,h)), M_\infty(K_d(R,h))\}.} \tag{20}$$

Equivalently, after identifying the axis with $\mathbb{R}$, the symmetry-reduced certified chest is exactly

$$\mathcal{C}_{\text{ax}}(K_d(R,h)) = [\, y_1(R,h), y_\infty(R,h)\,]$$

with the endpoints understood in either order. For $d=2$, this is precisely the reflection-axis reduction of Paper I's standard isosceles-triangle chest.

Proof. The convex-hull theorem gives $0 \le y \le h$.

For the first nontrivial bound, start from the regular simplex $K_d(R,h_*)$, whose center is forced to $g_* e$, and move only the apex along the axis from $h_* e$ to he. Attractivity gives

$$(h-h_*)(y-g_*) \ge 0. \tag{21}$$

For the second bound, pair the apex $A = he$ with one base vertex, say v_1. Let

$$u = A - v_1, \qquad m = \frac{A+v_1}{2},$$

and let $H = \{x : u\cdot(x-m) = 0\}$ be their perpendicular bisector. For every other base vertex v_j, $j \ge 2$,

$$u\cdot(v_j - m) = \frac{(d+1)R^2 - (d-1)h^2}{2(d-1)}. \tag{22}$$

Thus all remaining vertices lie on the same side of H, with the side changing exactly at $h = h_*$. Project them orthogonally to H. Reflection in H then swaps A, v_1 and fixes every projected point, so the predecessor center lies in H. Restoring the projected vertices uses parallel displacements and gives the useful-bisector inequality

$$\begin{cases} y \ge c(h), & h < h_*, \\ y \le c(h), & h > h_*, \end{cases} \qquad c(h) = \frac{h^2 - R^2}{2h}. \tag{23}$$

Indeed, for $x = ye$,

$$u\cdot(x-m) = h\,[y - c(h)].$$

If $h < h_*$, the three bounds are

$$0 \le y \le h, \qquad y \le g_*, \qquad y \ge c(h),$$

so

$$\max\{0, c(h)\} \le y \le \min\{h, g_*\},$$

which is exactly the interval between (18) and (19). If $h > h_*$, they become

$$g_* \le y \le c(h),$$

again the same interval. At $h = h_*$ symmetry collapses the interval to the single point $g_* e$. □

Remark 4.4 (Why this is an ambient theorem)**.** The bisector proof leaves the regular-base axial family at its intermediate projected configuration. Therefore Theorem 4.3 is a theorem about restrictions of *ambiently* attractive centers. It is not an outer theorem for arbitrary rules defined only on the kite stratum. This logical point will matter below.

4.4 Every power mean is constrained-attractive

For $p > 1$, put

$$M_p(K_d(R, h)) = y_p(R, h)e.$$

Strict convexity gives a unique minimizer. Its equilibrium equation is

$$\boxed{(h - y_p)^{p-1} = d\, y_p\, (R^2 + y_p^2)^{(p-2)/2}.} \tag{24}$$

Theorem 4.5 (Constrained attractivity of all power means)**.** *For every $d \ge 2$, every $p > 1$, and every regular-base axial simplex, M_p is orbit-attractive in the sense of Theorem 4.1. More precisely,*

$$0 < \frac{\partial y_p}{\partial h} = \frac{(p-1)(h - y_p)^{p-2}}{(p-1)(h - y_p)^{p-2} + d(R^2 + y_p^2)^{(p-4)/2}[R^2 + (p-1)y_p^2]} < 1. \tag{25}$$

Proof. Differentiate (24) with respect to h. The displayed formula follows immediately, and every term in its denominator is positive. The apex response is therefore positive. If the entire base plane is translated toward the apex while the apex is held fixed, the center height in absolute coordinates changes with derivative $1 - \partial_h y_p > 0$. Finally, a radial base change moves every base vertex in $e^\perp$, while symmetry keeps the center displacement on the axis, so the longitudinal dot product is zero. □

The exponent p itself parametrizes the whole variational interval.

Theorem 4.6 (Power sweep of the axial chest)**.** *For $h \ne h_*$, $p \mapsto y_p(R, h)$ is strictly monotone:*

$$\operatorname{sgn} \frac{\partial y_p}{\partial p} = \operatorname{sgn} \log \frac{h - y_p}{\sqrt{R^2 + y_p^2}} = \operatorname{sgn}(h - h_*). \tag{26}$$

At $h = h_*$, symmetry gives $y_p = g_*$ for every p. Moreover

$$\lim_{p \downarrow 1} M_p = M_1, \qquad \lim_{p \to \infty} M_p = M_\infty,$$

and consequently

$$\boxed{\overline{\{M_p(K_d(R, h)) : 1 < p < \infty\}} = \mathcal{C}_{\mathrm{ax}}(K_d(R, h)).} \tag{27}$$

Proof. Differentiate the logarithm of (24) with respect to p. The derivative with respect to y has the sign imposed by strict convexity, and one obtains the first equality in (26). The logarithm can change sign only when

$$h - y_p = \sqrt{R^2 + y_p^2}.$$

At such a point the two distance classes are equal, and (24) also requires

$$\sqrt{R^2 + y_p^2} = d\, y_p.$$

Eliminating y_p gives

$$(d-1)h^2 = (d+1)R^2,$$

that is, $h = h_*$. Hence for a nonregular kite the sign is constant in p. Evaluating at $p = 2$, where $y_2 = h/(d+1)$, shows that it is the sign of $h - h_*$. The endpoint limits are the standard limits of L^p-type Fréchet minimizers to the geometric median and the Chebyshev/minimum-ball center. Together with Theorems 4.2 and 4.3, this proves (27). ☐

Corollary 4.7 (Isosceles triangle formulas)**.** *For*

$$T_h = \{(0,h), (-1,0), (1,0)\},$$

one has $h_* = \sqrt{3}$, $g_* = 1/\sqrt{3}$, *and*

$$\mathcal{C}_{\mathrm{ax}}(T_h) = \operatorname{conv}\{M_1(T_h), M_\infty(T_h)\},$$

where

$$y_1(h) = \min\left\{h, \frac{1}{\sqrt{3}}\right\}, \qquad y_\infty(h) = \max\left\{0, \frac{h^2-1}{2h}\right\}.$$

Equivalently,

$$\mathcal{C}_{\mathrm{ax}}(T_h) = \begin{cases} [0,h], & 0 < h \le 1/\sqrt{3}, \\ [0, 1/\sqrt{3}], & 1/\sqrt{3} \le h \le 1, \\ [(h^2-1)/(2h), 1/\sqrt{3}], & 1 \le h \le \sqrt{3}, \\ [1/\sqrt{3}, (h^2-1)/(2h)], & h \ge \sqrt{3}. \end{cases}$$

For every $p > 1$, the constrained center $M_p(T_h)$ moves monotonically through this interval as p runs from 1 to ∞.

This is the closest two-dimensional analogue of the exact real-number theorem: the one-dimensional endpoints "median" and "midrange" are precisely M_1 and M_∞. The new obstruction is global extendability. Although every M_p is orbit-attractive on the isosceles stratum, only a proper exponent range is attractive on the full triangle domain: Paper III proves the exact interval $4 - 2\sqrt{2} \le p \le 4 + 2\sqrt{2}$ [2].

Conjecture 4.8 (Isosceles ambient BigHeart)**.** *For every non-equilateral isosceles triangle T_h,*

$$\boxed{\overline{\mathcal{V}^\omega(T_h)} \overset{?}{=} \mathcal{C}_{\mathrm{ax}}(T_h) = \operatorname{conv}\{M_1(T_h), M_\infty(T_h)\}.}$$

The analogous equality may hold for the C^∞ and C^1 closures.

The power-sweep theorem does *not* prove this conjecture, because the maps M_p for extreme p are not ambiently attractive. It does, however, identify a canonical target value for every point of the proposed heart and turns the isosceles problem into an extension problem: can each constrained power value be matched at the target by some globally attractive analytic center map?

Because $\mathcal{V}^\omega(T_h)$ is convex and contains the centroid, the conjecture is equivalent to the apparently stronger open-interval statement

$$\operatorname{relint}\mathcal{C}_{\mathrm{ax}}(T_h) \subseteq \mathcal{V}^\omega(T_h). \tag{28}$$

Indeed, a convex subset of a line whose closure is the entire closed interval must contain its relative interior. The next theorem shows that (28) is already true *locally in shape space*; what remains is global continuation.

4.5 Local PSD-admissible germs and the extension problem

Fix a non-equilateral isosceles triangle T_{h_0}, but now allow nearby triangles to become scalene. Since the base side is different in length from the two equal sides, on a sufficiently small permutation-saturated neighborhood of the isosceles orbit there is a well-defined analytic branch selecting the continuation of that base.

For such a nearby triangle let B, C be the selected base vertices and put

$$M = \frac{B+C}{2}, \qquad R = \frac{|B-C|}{2}, \qquad t = \frac{C-B}{|C-B|}.$$

Write the apex displacement from the base midpoint as

$$A - M = q\,t + h\,n,$$

where $n \perp t$, $h > 0$, and n points toward A. Thus

$$q = (A-M)\cdot t, \qquad h = \big|A - M - q\,t\big|, \qquad r = \frac{h}{R}.$$

Definition 4.9 (Locally PSD-admissible invariant germ)**.** A C^1 map defined on a permutation-saturated neighborhood U in triangle shape space is a *locally PSD-admissible invariant germ* if it is permutation invariant and Euclidean- and similarity-equivariant on U, and if all three symmetric vertex Jacobians are positive semidefinite wherever the corresponding infinitesimal vertex motion remains in U. This is a local first-order notion. Such a germ is not, merely by virtue of this definition, a member of $\mathcal{A}^1$ or $\mathcal{A}^\omega$, and the global convex-hull theorem need not apply to a prescribed value of the germ.

Theorem 4.10 (Local PSD-admissible realization at an isosceles orbit)**.** *Let ψ be real analytic near $r_0 = h_0/R_0$. Define the local center germ*

$$X_\psi = M + \psi'(r)\,q\,t + R\psi(r)\,n. \tag{29}$$

After taking the permutation-saturated union of the local base branches, X_ψ is permutation invariant, Euclidean equivariant, and similarity equivariant. At the isosceles target $q = 0$,

$$X_\psi(T_{h_0}) = M + R_0\psi(r_0)n. \tag{30}$$

If

$$s := \psi'(r_0),$$

then its symmetric vertex Jacobians at the target are

$$\boxed{S_A = sI_2, \qquad S_B = S_C = \frac{1-s}{2}I_2.} \tag{31}$$

Consequently, whenever $0 < s < 1$, the germ is strictly PSD-admissible on a full neighborhood of the isosceles orbit, including scalene perturbations.

Proof. The geometric quantities in (29) are intrinsic to the selected side and transform covariantly under Euclidean similarities. A swap of B, C reverses both t and q, leaving $q\,t$ unchanged, while M, R, h, n are unchanged. The permutation-saturated local definition therefore gives an invariant germ.

At the isosceles target choose coordinates

$$A = (0, h_0), \qquad B = (-R_0, 0), \qquad C = (R_0, 0).$$

With the base fixed, $q = A_x$, $h = A_y$, and $n = (0, 1)$ to first order. Differentiating (29) at $q = 0$ gives

$$J_A = sI_2.$$

Reflection in the axis fixes A and swaps B, C. Hence, if

$$J_B = \begin{pmatrix} p & u \\ v & w \end{pmatrix},$$

then

$$J_C = \begin{pmatrix} p & -u \\ -v & w \end{pmatrix}.$$

Translation equivariance, $J_A + J_B + J_C = I_2$, gives

$$p = w = \frac{1 - s}{2}.$$

The rotation and scaling identities

$$\sum_i J_i \Omega p_i = \Omega X, \qquad \sum_i J_i p_i = X$$

give respectively $u = (y - sh_0)/(2R_0)$ and $v = (sh_0 - y)/(2R_0) = -u$, where $y = R_0\psi(r_0)$. Thus the off-diagonal part of J_B is skew-symmetric and (31) follows. If $0 < s < 1$, all three symmetric blocks are positive definite. Strict positivity persists on a full neighborhood by continuity. □

Corollary 4.11 (No local first-order PSD value obstruction)**.** *At a non-equilateral isosceles triangle, every prescribed point on the symmetry axis is the value of a local analytic invariant germ whose symmetric vertex-Jacobian blocks are positive definite at the target. In particular, every point of* $\operatorname{relint} \mathcal{C}_{\mathrm{ax}}(T_h)$ *is locally realizable by a strictly PSD-admissible analytic invariant germ.*

Proof. Prescribe $\psi(r_0)$ to obtain the desired axial value in (30), and independently choose any $0 < \psi'(r_0) < 1$. An analytic function with the prescribed value and derivative exists, for example an affine function of r. □

Thus a failure of Theorem 4.8, if one exists, cannot be detected by a first-order calculation at the isosceles target. It must be a genuinely global obstruction: continuation around the full shape domain, compatibility with the other symmetry strata, or behavior near degenerate configurations.

4.6 Rotational lifts of one-dimensional regular means

There is a natural way to turn any ordinary real mean into an ambiently attractive vector center. It provides a large globally admissible class, but surprisingly it is still insufficient to solve the isosceles BigHeart problem.

Let $m : \mathbb{R}^N \to \mathbb{R}$ be permutation invariant, continuous, translation equivariant, positively homogeneous, reflection equivariant, and monotone in each coordinate. Write $\omega_{d-1} = |S^{d-1}|$.

Definition 4.12 (Rotational lift)**.** Define

$$\mathcal{R}_m(P) = \frac{d}{\omega_{d-1}} \int_{S^{d-1}} m(u \cdot p_1, \ldots, u \cdot p_N)\, u \, d\sigma(u). \tag{32}$$

Theorem 4.13 (Projection-lift attractivity)**.** *Every rotational lift $\mathcal{R}_m$ is a permutation-invariant, Euclidean- and similarity-equivariant attractive center on $(\mathbb{R}^d)^N$.*

Proof. The invariances follow from the corresponding scalar properties and a change of variables on the sphere. If only p_i is displaced by h, put

$$\delta(u) = m(\ldots, u \cdot (p_i + h), \ldots) - m(\ldots, u \cdot p_i, \ldots).$$

Coordinatewise monotonicity gives

$$\delta(u)\,(u \cdot h) \geq 0.$$

Therefore

$$h \cdot [\mathcal{R}_m(P_{i,h}) - \mathcal{R}_m(P)] = \frac{d}{\omega_{d-1}} \int_{S^{d-1}} \delta(u)(u \cdot h)\, d\sigma(u) \geq 0.$$

□

For three real inputs the exact theorem of Paper I says that every such mean lies between the median and the midrange. One might therefore hope that rotational lifting of scalar means fills the isosceles vector chest. It does not.

Let $\mathcal{R}_{\text{med}}$ and $\mathcal{R}_{\text{mid}}$ denote the lifts of the scalar median and midrange. The latter is precisely the classical Steiner point of the triangle [10]. Put

$$\alpha(h) = \arctan \frac{1}{h}.$$

Theorem 4.14 (Projection-lift obstruction)**.** *For $T_h = \{(0, h), (-1, 0), (1, 0)\}$, the entire class of rotational lifts of real regular three-variable means has axial attainable interval*

$$\boxed{[\mathcal{R}_{\text{med}}(T_h), \mathcal{R}_{\text{mid}}(T_h)]} \tag{33}$$

with the endpoints understood in either order. Their heights are

$$y_{\text{med}}(h) = \frac{2h}{\pi} \arctan \frac{1}{h}, \tag{34}$$

$$y_{\text{mid}}(h) = h \left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{1}{h} \right). \tag{35}$$

They satisfy

$$y_{\text{med}} + 2y_{\text{mid}} = h, \tag{36}$$

and coincide only at $h = \sqrt{3}$. For every non-equilateral isosceles triangle both lie strictly in the relative interior of $\mathcal{C}_{\text{ax}}(T_h)$. Hence rotational projection averaging cannot reach either variational endpoint $M_1(T_h)$ or $M_\infty(T_h)$.

Proof. The explicit formulas follow by integrating over directions $u = (\cos\theta, \sin\theta)$. On the upper semicircle the three projections are

$$h \sin\theta, \qquad -\cos\theta, \qquad \cos\theta.$$

The change in ordering occurs at $\theta = \alpha(h)$. A direct integration gives (34) and (35); (36) is also the rotational lift of the scalar identity med $+2$ midrange $= x_1 + x_2 + x_3$.

For the extremal statement, write a scalar invariant mean on an ordered triple $x \le y \le z$ as

$$m(x,y,z) = x + (z-x)f(t), \qquad t = \frac{y-x}{z-x}.$$

Reflection equivariance gives $f(1-t) = 1 - f(t)$, and the exact one-dimensional chest theorem gives, on $0 \le t \le 1/2$,

$$t \le f(t) \le \frac{1}{2}.$$

After splitting the directional integral according to the three possible projection orderings and transporting all pieces to $0 \le t \le 1/2$, the coefficient of $f(t)$ has the constant sign

$$\operatorname{sgn}(h^2 - 3).$$

The explicit positive factorization is recorded in Section A. The two extrema are therefore attained by $f(t) = t$ and $f(t) = 1/2$, i.e. by the median and midrange. Convex mixing fills the interval between their lifts. Direct comparison of (34)–(35) with Theorem 4.7 shows that the inclusion in the axial chest is strict unless $h = \sqrt{3}$. □

The preceding two theorems isolate the remaining difficulty sharply. There are many globally attractive ambient constructions, but the most canonical one inherited from the real-number theorem is too rigid. On the other hand, every desired isosceles value has a strictly PSD-admissible local invariant germ. The BigHeart problem is therefore a *global extension problem* between these two facts.

4.7 Sharp boundary asymptotics on the isosceles ray

The symmetry-reduced chest has a particularly simple consequence at the two ends of the isosceles shape ray. Write

$$T_h = \{(-1,0),(1,0),(0,h)\}, \qquad X(T_h) = (0, Y(h)).$$

The quotient

$$\tau_X(h) = \frac{Y(h)}{h}$$

is exactly the normalized barycentric weight of the apex whenever barycentric coordinates are used.

Theorem 4.15 (Sharp asymptotic response bounds)**.** *Let X be any ambiently attractive invariant center for which the indicated limits exist. Then*

$$0 \le \lim_{h \downarrow 0} \frac{Y(h)}{h} \le 1, \tag{37}$$

$$0 \le \lim_{h \to \infty} \frac{Y(h)}{h} \le \frac{1}{2}. \tag{38}$$

All four endpoint values $0, 1, 0, 1/2$ *are sharp in the class of globally attractive continuous centers.*

Proof. For $0 < h \le 1/\sqrt{3}$, Theorem 4.7 gives the exact axial chest $0 \le Y(h) \le h$, hence (37). For $h \ge \sqrt{3}$ it gives

$$\frac{1}{\sqrt{3}} \le Y(h) \le \frac{h^2-1}{2h},$$

so

$$\frac{1}{\sqrt{3}h} \le \frac{Y(h)}{h} \le \frac{1}{2} - \frac{1}{2h^2},$$

which proves (38).

Sharpness follows already from the two rotational lifts in Theorem 4.14. Their normalized heights are

$$\frac{y_{\mathrm{med}}(h)}{h} = \frac{2}{\pi}\arctan\frac{1}{h}, \qquad \frac{y_{\mathrm{mid}}(h)}{h} = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{1}{h}.$$

Thus the median lift has endpoint signature $(1, 0)$ and the midrange/Steiner lift has signature $(0, 1/2)$. ☐

There is also a direct one-dimensional interpretation of the tall limit. After scaling T_h by $1/h$, the point multiset tends to $(0, 0, 1)$ on a line, while similarity equivariance sends the center height to $Y(h)/h$. The exact real-number theorem of Paper I therefore predicts precisely the interval between the median 0 and the midrange 1/2. The flat limit is instead a transverse response at the collinear configuration $\{(-1, 0), (1, 0), (0, 0)\}$: lifting the middle point gives $Y \geq 0$, while the translation comparison obtained by lowering the two base points gives $Y \leq h$.

4.7.1 Center-function signatures at the two degenerate ends

Suppose a Kimberling center function produces cyclic barycentric weights

$$f(a, b, c) : f(b, c, a) : f(c, a, b), \qquad f(a, b, c) = f(a, c, b),$$

and put $s = \sqrt{1 + h^2}$. On T_h the normalized apex weight is exactly

$$\boxed{\frac{Y(h)}{h} = \frac{f(2, s, s)}{2f(s, s, 2) + f(2, s, s)}.} \tag{39}$$

Consequently, whenever the displayed boundary values are finite and the denominator has a nonzero leading term,

$$\lim_{h\downarrow 0} \frac{Y(h)}{h} = \frac{f(2, 1, 1)}{2f(1, 1, 2) + f(2, 1, 1)}, \tag{40}$$

$$\lim_{h\to\infty} \frac{Y(h)}{h} = \frac{f(0, 1, 1)}{2f(1, 1, 0) + f(0, 1, 1)}. \tag{41}$$

Here homogeneity is used in the second limit. Thus the generic algebraic mechanisms behind the four sharp endpoint classes are

asymptotic class	generic center-function condition
$h \downarrow 0,\ Y/h \to 0$	$f(2, 1, 1) = 0$
$h \downarrow 0,\ Y/h \to 1$	$f(1, 1, 2) = 0$
$h \to \infty,\ Y/h \to 0$	$f(0, 1, 1) = 0$
$h \to \infty,\ Y/h \to 1/2$	$f(0, 1, 1) = 2f(1, 1, 0).$

If numerator and denominator vanish simultaneously, the leading asymptotic orders must be compared instead; this is exactly the situation in which very slow numerical convergence can occur.

4.7.2 Numerical boundary classes among ETC survivors

The archived exploratory retest contains 2, 428 distinct indices and 632 formulas not refuted by its Jacobian screen. The accompanying endpoint table records 38 finite-height values at $h = 10^{-6}$ outside $[0, 1]$, leaving 594 unrefuted numerical candidates. These are numerical witnesses, not exact sign certificates; arbitrary-precision evaluation alone supplies no rigorous rounding-error enclosure. The supplement includes the historical tables, their complete index audit, a pinned local ETC snapshot for new runs, and scripts for the scan and selected slow cases. The historical generating snapshot was not recorded, so equality of that snapshot with the bundled one is not asserted. The stored tables are independently auditable without it.

The reported fitted endpoint classes have counts 55, 9, 143, and 53 for the signatures $h \downarrow 0 : Y/h \to 0, 1$ and $h \to \infty : Y/h \to 0, 1/2$, respectively. These are exploratory classifications, not proved limits. The count 9 includes the manually annotated slow cases $X(506), X(507)$; their historical direct-check transcript was not supplied. The revised list generator distinguishes fitted classifications from these annotations and never treats the latter as certified limits. A useful future invariant of a candidate center is therefore its joint endpoint signature

$$\left(\lim_{h\downarrow 0} \frac{Y(h)}{h}, \lim_{h\to\infty} \frac{Y(h)}{h}\right).$$

4.8 Complex shape coordinates and a final extension attempt

The global extension problem admits a compact formulation in one complex shape variable. Identify the plane with $\mathbb{C}$ and normalize an oriented triangle by

$$A = 0, \qquad B = 1, \qquad C = z, \qquad \Im z > 0.$$

Write the normalized center as

$$X(0, 1, z) = f(z, \bar{z}).$$

For a real-linear map written in complex form as

$$h \longmapsto ah + b\bar{h},$$

the eigenvalues of its symmetric part are $\Re a \pm |b|$. Hence its symmetric part is positive semidefinite if and only if

$$\Re a \geq |b|. \tag{42}$$

Theorem 4.16 (Exact complex cone formulation)**.** *A C^1 invariant center is ambiently attractive exactly when its normalized shape function satisfies, throughout the upper half-plane,*

$$\Re f_z \geq |f_{\bar{z}}|, \tag{43}$$
$$\Re(f - zf_z) \geq |z|\,|f_{\bar{z}}|, \tag{44}$$
$$\Re(1 - f + (z-1)f_z) \geq |z-1|\,|f_{\bar{z}}|. \tag{45}$$

The three left-hand sides add to 1, and therefore every attractive center also obeys the global anti-holomorphic budget

$$\boxed{(1 + |z| + |z-1|)\,|f_{\bar{z}}| \leq 1.} \tag{46}$$

Proof. For motion of C, the differential is

$$\delta X = f_z h + f_{\bar{z}}\bar{h}.$$

If B moves while A, C are held fixed, renormalizing the ordered triangle gives

$$\delta X = (f - zf_z)h - \bar{z}f_{\bar{z}}\bar{h}.$$

Similarly, motion of A gives

$$\delta X = (1 - f + (z-1)f_z)\,h + (\bar{z}-1)f_{\bar{z}}\bar{h}.$$

Apply (42) to these three real-linear maps. Summing the three resulting left sides proves (46). $\square$

Permutation invariance and reflection equivariance become the functional relations

$$f(1-z)=1-f(z), \qquad f(1/z)=\frac{f(z)}{z}, \qquad f(\bar{z})=\overline{f(z)}, \tag{47}$$

with the remaining permutations generated by the first two transformations. The isosceles family is the fixed line

$$z=\frac{1}{2}+it, \qquad t>0.$$

On this line (47) forces

$$f\left(\frac{1}{2}+it\right)=\frac{1}{2}+iq(t), \qquad \frac{Y(h)}{h}=\frac{q(t)}{t}, \qquad h=2t. \tag{48}$$

Thus the BigHeart conjecture may be read as an S_3-equivariant first-order extension problem for a complex map satisfying the three cone inequalities.

Here holomorphic refers to complex differentiability, whereas analytic centers in $\mathcal{A}^\omega$ are real analytic. These regularity notions must not be identified.

The local models of Theorem 4.10 are especially simple in this language: after a side has been locally distinguished, their derivative can be chosen with $f_{\bar{z}}=0$ at the target. It is therefore natural to ask whether one can globalize the construction while remaining holomorphic. The answer is rigidly negative.

Theorem 4.17 (Holomorphic rigidity)**.** *Assume attractivity and the standing continuous extension to degenerate configurations. If a permutation-invariant Euclidean- and similarity-equivariant center has $f_{\bar{z}}\equiv 0$ on the upper half-plane, then*

$$\boxed{f(z)=\frac{1+z}{3};} \tag{49}$$

i.e. the center is the centroid.

Proof. Convex-hull internality makes $f(x)$ real for every non-collision collinear shape $x\in\mathbb{R}$. Reflection equivariance and Schwarz reflection extend f holomorphically across the real axis; the possible singularities at $0,1$ are removable because internality keeps f bounded there. Moreover $f(z)\in\operatorname{conv}\{0,1,z\}$ implies

$$|f(z)|\le\max\{1,|z|\},$$

so the resulting entire function has at most linear growth and hence is affine, $f(z)=az+b$. The two permutation equations in (47) give $a=b$ and $a+2b=1$, whence $a=b=1/3$. $\square$

Thus every non-centroid global attractive center must use a nonzero anti-holomorphic component somewhere, even though a perfect local isosceles germ may be holomorphic to first order. The budget (46) quantifies how much of that component is available. This is the point at which the complex proof attempt stops: we have not found a global S_3-equivariant solution of (43)–(45) whose prescribed isosceles value approaches an arbitrary variational endpoint. The reformulation is nevertheless useful because it reduces six real vertex derivatives to one complex first-order differential inclusion and shows precisely why a purely holomorphic continuation of the local construction cannot work.

4.9 Endpoint non-attainment and the conjectured closure

The preceding analysis also clarifies the correct formulation of the isosceles BigHeart conjecture. Its endpoints should not be expected to be values of one analytic attractive center.

Lemma 4.18 (Equality propagation along a displacement certificate)**.** *Let X be a C^1 globally attractive center and let $P(t)$, $0 \le t \le 1$, move only the ith input with constant velocity d. If*

$$d \cdot \big[X(P(1)) - X(P(0))\big] = 0,$$

then

$$d^T S_i(P(t))d = 0 \qquad (0 \le t \le 1).$$

In particular $S_i(P(1))d = 0$.

Proof. Along the path,

$$\frac{d}{dt}\, d \cdot X(P(t)) = d^T J_i(P(t))d = d^T S_i(P(t))d \ge 0.$$

The integral of this continuous nonnegative function is the assumed zero endpoint difference, hence the integrand vanishes identically. For a positive semidefinite matrix, $d^T S_i d = 0$ implies $S_i d = 0$. □

Lemma 4.19 (Reflection-symmetric first jet)**.** *At*

$$A = (0, h), \qquad B = (-1, 0), \qquad C = (1, 0), \qquad X(T_h) = (0, Y(h)),$$

let $b = Y'(h)$. Every C^1 invariant center has, for some real α,

$$S_A = \begin{pmatrix} \alpha & 0 \\ 0 & b \end{pmatrix}, \qquad S_C = \begin{pmatrix} \dfrac{1-\alpha}{2} & -\dfrac{h(b-\alpha)}{4} \\ -\dfrac{h(b-\alpha)}{4} & \dfrac{1-b}{2} \end{pmatrix},$$

and S_B is its reflected counterpart.

Proof. Reflection in the vertical axis fixes A and interchanges B, C, forcing S_A to be diagonal and relating the two base-vertex blocks. The vertical entry of S_A is the derivative of the axial center value under vertical motion of the apex, namely $b = Y'(h)$. Translation equivariance gives $J_A + J_B + J_C = I$. The infinitesimal scaling and rotation identities

$$\sum_i J_i p_i = X, \qquad \sum_i J_i \Omega p_i = \Omega X$$

then determine the symmetric off-diagonal entry and the remaining diagonal entries, yielding the displayed matrix. No attractivity assumption is used in this kinematic calculation. □

Theorem 4.20 (Non-attainment of the isosceles chest endpoints)**.** *Let $X \in \mathcal{A}^\omega$ and write $X(T_h) = (0, Y(h))$. If $h \ne \sqrt{3}$, then $Y(h)$ lies strictly between the two endpoints $y_1(h)$ and $y_\infty(h)$ of Theorem 4.7. In particular the isosceles BigHeart conjecture, if true, is necessarily a statement about $\overline{\mathcal{V}^\omega(T_h)}$, not about the unclosed analytic attainable set.*

Proof. Orbit-attractivity gives $0 \le Y'(h) \le 1$, and symmetry gives $Y(\sqrt{3}) = 1/\sqrt{3}$.

First consider the straight boundary pieces $Y = 0$, $Y = h$, and $Y = 1/\sqrt{3}$. If an analytic Y touches 0, monotonicity forces it to vanish on a nontrivial adjacent interval. If it touches h, the nonnegative monotone function $h - Y(h)$ vanishes on a nontrivial interval. If it touches $1/\sqrt{3}$ away from the equilateral shape, monotonicity together with the chest inequality forces constancy on the interval between the contact point and $h = \sqrt{3}$. Analytic continuation would make the corresponding identity global, contradicting the other symmetry-forced or chest values.

It remains to exclude contact with

$$c(h) = \frac{h^2 - 1}{2h}, \qquad h > 1.$$

Use the labelling of Theorem 4.19. The Paper-I useful-bisector certificate is obtained by projecting the opposite base vertex to the perpendicular bisector and restoring it along a direction parallel to $d = B - A = (-1, -h)$. If the final support inequality is an equality, Theorem 4.18 gives

$$d^T S_C d = 0, \qquad \text{hence} \qquad S_C (1, h)^T = 0,$$

where the sign of the null vector is immaterial. Substituting the matrix from Theorem 4.19, the two scalar equations give

$$\alpha = 3b - 2, \qquad (1 - b)(3 - h^2) = 0.$$

At a non-equilateral isosceles triangle $h \neq \sqrt{3}$, so contact with the curved certificate boundary forces $Y'(h) = b = 1$.

On the other hand, the globally attractive value function lies locally on one side of $c(h)$. A differentiable function touching a differentiable boundary from one side has the same first derivative there, so

$$Y'(h) = c'(h) = \frac{1}{2}\left(1 + \frac{1}{h^2}\right) < 1 \qquad (h > 1),$$

a contradiction. □

The theorem makes the sharpened open-interval formulation (28) the natural target: prove that every strict chest value is globally attainable analytically, and obtain the two variational endpoints only by taking limits of globally attractive analytic centers.

5 Open problems

The results of this paper reduce the isosceles Heart problem to global extension. The principal remaining questions are:

1. determine whether every interior point of the symmetry-reduced axial chest is attained by a globally attractive analytic triangle center;
2. construct global solutions of the complex first-order cone system with a genuinely nonzero anti-holomorphic component;
3. characterize the endpoint signatures of globally attractive centers and decide which signatures can coexist in one analytic center map;
4. extend the axial-simplex analysis to other fixed-point strata of finite reflection groups; and
5. extend the supplied versioned supplement by certifying saved witness signs and proving asymptotic signatures symbolically.

6 Conclusion

The isosceles stratum is an exact meeting point of one-dimensional mean theory and planar triangle-center geometry. Reflection symmetry reduces every invariant value to one axial coordinate. Finite ambient displacement arguments give a canonical certificate chest, the interval from the geometric median to the minimum-enclosing-ball center, while the constrained power family sweeps the whole interval. The same statement holds for every regular-base axial simplex in every dimension.

The essential obstruction is global rather than local. Every prescribed interior axial value possesses a strictly PSD-admissible analytic invariant germ, yet the most canonical globally attractive rotational lifts fill only a strict subinterval. The two degenerate ends of the isosceles ray have sharp normalized response intervals $[0,1]$ and $[0,1/2]$. Complex shape coordinates express ambient attractivity as three scalar first-order cone inequalities coupled by the permutation group, and holomorphic rigidity shows that a nontrivial global extension must use an anti-holomorphic component.

Thus the isosceles Heart problem has been converted into a precise analytic continuation problem: reconcile flexible local PSD germs with global permutation covariance and finite displacement monotonicity. The companion Paper IV develops the complementary global heart, jet, and semidefinite machinery for arbitrary triangles [3].

A Projection-lift kernel on an isosceles triangle

We record the sign computation used in Theorem 4.14. Let

$$T_h = \{(0,h), (-1,0), (1,0)\},$$

and parameterize upper-half-plane directions by $u = (\cos\theta, \sin\theta)$. After exploiting antipodal symmetry, split the integral according to the ordering of

$$h\sin\theta, \qquad -\cos\theta, \qquad \cos\theta.$$

Write $r = h\tan\theta$, then use reflection of the scalar mean to transport all three ordering ranges to a common parameter $0 \le t \le 1/2$. If

$$m(x,y,z) = x + (z-x) f\left(\frac{y-x}{z-x}\right) \qquad (x \le y \le z),$$

the coefficient of $f(t)$ in the resulting axial integral is

$$(h^2-3)\,\Xi_h(t), \tag{50}$$

where

$$\Xi_h(t) = \frac{4h^2(2-t)(1+t)(1-2t)\,P_h(t)}{(h^2+4t^2-4t+1)^2(h^2t^2+t^2-4t+4)^2(h^2(1-t)^2+(1+t)^2)^2}.$$

Put $u = t(1-t) \in [0,1/4]$. The numerator polynomial is

$$P_h(t) = h^6 P_3(u) + h^4 P_2(u) + h^2 P_1(u) + P_0(u),$$

with

$$\begin{aligned}
P_3(u) &= 1 - 3u + 2u^2 - u^3,\\
P_2(u) &= 7 - 21u + 30u^2 - 7u^3,\\
P_1(u) &= 11 - 33u + 6u^2 - 11u^3,\\
P_0(u) &= 5 - 15u + 42u^2 - 5u^3.
\end{aligned}$$

Each is strictly positive on $0 \le u \le 1/4$. For example, discarding the positive quadratic terms gives the crude lower bounds

$$P_3 \ge 1 - \frac{3}{4} - \frac{1}{64} > 0, \quad P_2 \ge 7 - \frac{21}{4} - \frac{7}{64} > 0,$$

$$P_1 \ge 11 - \frac{33}{4} - \frac{11}{64} > 0, \quad P_0 \ge 5 - \frac{15}{4} - \frac{5}{64} > 0.$$

All denominator factors in Ξ_h are positive, and $(2-t)(1+t)(1-2t) \ge 0$ on the integration interval. Thus $\Xi_h(t) > 0$ in its interior, proving that the full coefficient has the constant sign of $h^2 - 3$.

Since the exact real three-point theorem gives

$$t \le f(t) \le \frac{1}{2} \qquad (0 \le t \le 1/2),$$

the rotational-lift support is extremized globally by $f(t) = t$ and $f(t) = 1/2$, namely the scalar median and midrange. This completes the factorization argument used in Theorem 4.14.

References

[1] I. Hammoudeh, *Attractive Centers I: Chests, Hearts, and Symmetry–Displacement Geometry*, companion manuscript, 2026.

[2] I. Hammoudeh, *Attractive Centers III: Radial Location Functionals, Mechanical Response, and an Exact Elasticity Classification*, companion manuscript, 2026.

[3] I. Hammoudeh, *Attractive Centers IV: Analytic Triangle Hearts, Centroidal Rays, and Nonlocal Constraints*, companion manuscript, 2026.

[4] C. Kimberling, *Triangle Centers and Central Triangles*, Congressus Numerantium 129, Utilitas Mathematica Publishing, 1998.

[5] C. Kimberling, *Encyclopedia of Triangle Centers*, University of Evansville, https://faculty.evansville.edu/ck6/encyclopedia/, accessed September 2026.

[6] R. Pérez-Fernández, B. De Baets, and M. Gagolewski, A taxonomy of monotonicity properties for the aggregation of multidimensional data, *Information Fusion* **52** (2019), 322–334.

[7] M. Fréchet, Les èléments aléatoires de nature quelconque dans un espace distancié, *Annales de l'Institut Henri Poincaré* **10** (1948), 215–310.

[8] E. Weiszfeld, Sur le point pour lequel la somme des distances de n points donnés est minimum, *Tohoku Mathematical Journal* **43** (1937), 355–386.

[9] E. Welzl, Smallest enclosing disks (balls and ellipsoids), in H. Maurer (ed.), *New Results and New Trends in Computer Science*, Lecture Notes in Computer Science 555, Springer, 1991, 359–370.

[10] G. C. Shephard, The Steiner point of a convex polytope, *Canadian Journal of Mathematics* **18** (1966), 1294–1300.

[11] A. Lou, I. Katsman, Q. Jiang, S. Belongie, S.-N. Lim, and C. De Sa, Differentiating through the Fréchet mean, *Proceedings of the 37th International Conference on Machine Learning*, PMLR **119** (2020), 6393–6403.

[12] B. Sturmfels, *Algorithms in Invariant Theory*, 2nd ed., Springer, Vienna, 2008.

[13] S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, 2nd ed., Birkhäuser, Boston, 2002.

[14] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.