Attractive Centers III

Radial Location Functionals, Mechanical Response, and an Exact Elasticity Classification

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September 2026; revised submission version

Abstract

Papers I and II study *invariant attractive centers*: permutation-invariant, Euclidean-equivariant, positively scale-equivariant rules satisfying a finite single-input displacement inequality. The present paper enlarges the setting to radial variational *location functionals*

$$X_\psi(P) = \arg\min_x \sum_i \psi(\|x - p_i\|),$$

which are always permutation- and Euclidean-equivariant but need not be scale equivariant. We first identify the intersection with the original center class: under the standing smoothness and strict-convexity hypotheses, X_ψ is similarity-equivariant for every finite configuration if and only if $\psi'(r) = Cr^k$, hence, up to an irrelevant additive constant, $\psi(r) = Cr^{k+1}/(k+1)$.

For the broader radial class we derive the exact response identity

$$J_i = K^{-1}K_i, \qquad K_i = \frac{\psi'(r_i)}{r_i}\big[I + (\eta(r_i) - 1)e_i e_i^T\big], \qquad \eta(r) = \frac{r\psi''(r)}{\psi'(r)}.$$

The quantity η is the ratio of radial to tangential stiffness. Using positive-operator turning angles and a fixed-radius planar obstruction, we prove the sharp classification

$$X_\psi \text{ satisfies the global finite displacement law} \iff 3 - 2\sqrt{2} \le \eta(r) \le 3 + 2\sqrt{2} \quad (r > 0).$$

The collision case is obtained from a log-force regularization that preserves the elasticity band and is quadratic near the origin. Thus one out-of-band radius already forces a finite planar counterexample, whereas strict pointwise inequalities give positive-definite response at every fixed finite regular configuration.

For the similarity-invariant subfamily this yields the exact theorem

$$M_p \text{ is an attractive center} \iff 4 - 2\sqrt{2} \le p \le 4 + 2\sqrt{2},$$

and outside the interval noncollinear three-point planar counterexamples suffice. We also formulate the corresponding continuum response for push-forward material perturbations, prove centerwise consistency and Hausdorff convergence of the convex power-generated region under weak compact empirical approximation, and contrast radial equilibrium with skeletal mass centers, whose intermediate tetrahedral examples fail attractivity. The paper therefore separates scale-free attractive centers from the larger mechanically natural class of attractive radial location functionals while giving an exact stiffness criterion for both.

Contents

1 Introduction

The theory of attractive centers began with a replacement for monotonicity in vector-valued averaging. If one datum p_i is displaced by h, an attractive center X satisfies

$$h \cdot \big[X(P_{i,h}) - X(P)\big] \geq 0. \tag{1}$$

For a C^1 center this is equivalent to

$$\operatorname{Sym} D_{p_i} X \succeq 0$$

for every input and every configuration. Paper I developed finite symmetry–displacement chests; Papers II and IV study isosceles extension and analytic triangle hearts, using power-distance centers as a source of certified inner values [1, 3].

The variational family deserves a theory of its own. Fréchet's metric-space notion of a mean is based on minimizing a distance cost [4]. For Euclidean data the geometric median is the classical Fermat–Weber location problem and is associated with Weiszfeld's algorithm [5, 6]. Modern computational work treats general power means and their coresets [7]. In robust statistics, radial losses are a standard way to interpolate between quadratic and more outlier-resistant objectives [8]. None of these viewpoints, by itself, answers the present question: when does moving one input in a direction force the minimizing location to have a nonnegative response in that same direction?

The mechanical interpretation makes the issue particularly transparent. Imagine a massless node at x joined to anchors p_i by identical isotropic interactions with potential $\psi(r)$. The minimizing location is the stable equilibrium of this star. Moving one anchor changes both the load and the local stiffness. The response is attractive in the *energy metric* automatically, but Euclidean attractivity can fail because two positive stiffness operators need not have a positive symmetric product. This is a matrix-angle problem of the type studied in Kantorovich–Wielandt and antieigenvalue theory [10, 11].

The principal results are:

1. a precise separation between the original similarity-invariant center class and the broader class of Euclidean radial location functionals;
2. a scale-rigidity theorem showing that the similarity-equivariant radial members are exactly the pure power laws;
3. a general stiffness identity and an exact pointwise elasticity classification for the *global finite* displacement law;
4. a fixed-radius planar obstruction proving necessity, together with an exact power-growth characterization of all admissible force laws;
5. an exact three-point if-and-only-if theorem for the power-distance centers;
6. a continuum push-forward response theorem and centerwise empirical consistency; and
7. a comparison with skeletal mass centers, showing that physically natural positive-mass constructions need not be attractive.

2 Radial variational location functionals

Let

$$P = (p_1, \dots, p_N) \in (\mathbb{R}^d)^N$$

and let $\psi : [0, \infty) \to \mathbb{R}$ be increasing and strictly convex, with $\psi \in C^2(0, \infty)$, $\psi'(r) > 0$, and $\psi''(r) > 0$ for $r > 0$. Assume also that $\psi(r) \to \infty$ as $r \to \infty$.

Definition 2.1 (Radial variational location functional)**.** The radial variational location functional associated with ψ is

$$X_\psi(P) = \arg\min_{x\in\mathbb{R}^d} \Phi_\psi(x;P), \qquad \Phi_\psi(x;P) = \sum_{i=1}^{N} \psi(\|x-p_i\|). \tag{2}$$

Strict convexity gives uniqueness. Every X_ψ is permutation invariant and equivariant under translations and orthogonal transformations. It need *not* be positively scale equivariant: a fixed nonhomogeneous potential introduces an intrinsic length scale. This is a genuine enlargement of the class used in Papers I and II, where an *invariant center* additionally satisfies

$$X(sP) = sX(P) \qquad (s>0).$$

Accordingly, throughout this paper *radial location functional* refers to the broader class, while *attractive center* is reserved for a rule that also satisfies the scale-equivariance axiom of the earlier papers.

We work first at a *regular configuration*, meaning that the minimizer $x = X_\psi(P)$ does not coincide with an input point. Collisions are treated later by an elasticity-preserving regularization.

Put

$$u_i = x - p_i, \qquad r_i = \|u_i\|, \qquad e_i = u_i/r_i,$$

and define the radial force magnitude

$$F(r) = \psi'(r).$$

The stationarity equation is

$$\sum_{i=1}^{N} F(r_i)e_i = 0. \tag{3}$$

Equivalently, with

$$a(r) = \frac{F(r)}{r} = \frac{\psi'(r)}{r} > 0, \tag{4}$$

one has

$$\sum_i a(r_i)(x-p_i) = 0, \qquad x = \frac{\sum_i a(r_i)p_i}{\sum_i a(r_i)}. \tag{5}$$

Thus the variational location is a positive, self-consistently weighted barycenter and is automatically internal.

2.1 Scale rigidity: when a radial location functional is a center

Theorem 2.2 (Scale rigidity of radial variational rules)**.** *Under the standing assumptions, the following are equivalent:*

(i) $X_\psi(sP) = sX_\psi(P)$ *for every finite configuration* P *and every* $s>0$*;*

(ii) there exist constants $C>0$ *and* $k>0$ *such that*

$$F(r) = \psi'(r) = Cr^k \qquad (r>0);$$

(iii) up to an additive constant,

$$\psi(r) = \frac{C}{k+1} r^{k+1}.$$

Thus the similarity-invariant members of the radial variational class are precisely the power-distance centers.

Proof. The implications (ii)$\Leftrightarrow$(iii) and (ii)$\Rightarrow$(i) are immediate. Assume (i). Consider on a line a multiset consisting of m copies of one endpoint and n copies of the other. If the minimizing point has distances $a, b > 0$ from the two clusters, stationarity is

$$mF(a) = nF(b). \tag{6}$$

Scale equivariance requires the scaled point to remain the minimizer after multiplication of all distances by s, hence

$$mF(sa) = nF(sb).$$

Therefore

$$\frac{F(sa)}{F(sb)} = \frac{F(a)}{F(b)} \tag{7}$$

whenever $F(b)/F(a) = m/n$ is rational.

Fix arbitrary $a, b > 0$. Since $F' > 0$, F is continuous and strictly increasing. Choose positive rationals $q_j \to F(b)/F(a)$ sufficiently close that $q_j F(a)$ belongs to the range of F, and let $b_j = F^{-1}(q_j F(a))$. Then $b_j \to b$, and applying (7) to a, b_j, followed by continuity, gives

$$\frac{F(sa)}{F(sb)} = \frac{F(a)}{F(b)}$$

for arbitrary $a, b, s > 0$. Hence $F(sr)/F(r) = c(s)$ is independent of r. The positive continuous function c satisfies $c(st) = c(s)c(t)$, so $c(s) = s^k$ for some real k. Since F is strictly increasing, $k > 0$. Taking $r = 1$ gives $F(r) = F(1)r^k$. □

Remark 2.3. The proof uses multiplicities only as a convenient way to realize rational force ratios. If configurations are required to have distinct inputs, split each cluster into sufficiently small symmetric clouds and pass to the limit.

2.2 Radial and tangential stiffness

Differentiate the force vector

$$F(r)e = \frac{F(r)}{r}u.$$

The resulting individual stiffness is

$$K_i = F'(r_i)e_i e_i^T + \frac{F(r_i)}{r_i}(I - e_i e_i^T). \tag{8}$$

Introduce the dimensionless logarithmic force elasticity

$$\boxed{\eta(r) = \frac{rF'(r)}{F(r)} = \frac{r\psi''(r)}{\psi'(r)} = \frac{\mathrm{d}\log F}{\mathrm{d}\log r}.} \tag{9}$$

Then

$$\boxed{K_i = a(r_i)\big[I + (\eta(r_i) - 1)e_i e_i^T\big].} \tag{10}$$

The tangential stiffness is $a(r_i)$; the radial stiffness is $a(r_i)\eta(r_i) = F'(r_i)$. Hence η is exactly the ratio

$$\frac{\text{radial stiffness}}{\text{tangential stiffness}}.$$

Let

$$K = \sum_i K_i = D_x^2 \Phi_\psi(x; P). \tag{11}$$

Our assumptions imply $K_i \succ 0$ and $K \succ 0$ at every regular configuration.

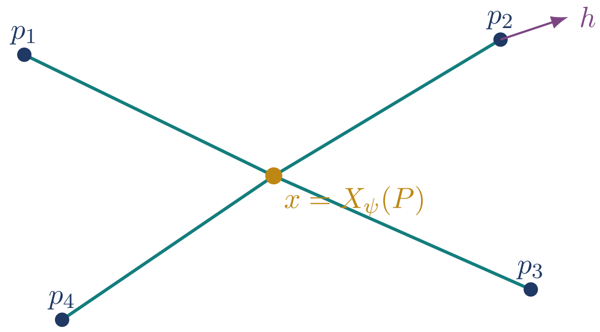

Figure 1: Mechanical interpretation of a radial variational location functional. The response to moving one anchor is the stiffness partition $J_i = K^{-1}K_i$.

Proposition 2.4 (Stiffness response formula). *If only p_i is displaced infinitesimally, then*

$$\boxed{J_i := D_{p_i} X_\psi = K^{-1}K_i.} \tag{12}$$

In particular,

$$\sum_i J_i = I. \tag{13}$$

Proof. Differentiate the equilibrium equation with respect to the minimizer and the moved anchor. A change δx contributes $K\delta x$; moving p_i by h contributes $-K_i h$. Thus $K\delta x = K_i h$, proving (12). Summing over i gives (13). □

3 Energy-metric positivity versus Euclidean attractivity

Formula (12) contains an important subtlety. Both K and K_i are positive definite, but their product need not have positive Euclidean symmetric part.

Proposition 3.1 (Natural-metric positivity). *Equip $\mathbb{R}^d$ with the energy inner product*

$$\langle u, v\rangle_K = u^T K v.$$

Then every response operator J_i is self-adjoint and positive in this metric:

$$\langle u, J_i v\rangle_K = u^T K_i v = \langle J_i u, v\rangle_K, \qquad \langle h, J_i h\rangle_K = h^T K_i h > 0. \tag{14}$$

Moreover J_i is similar to the symmetric positive matrix

$$K^{-1/2} K_i K^{-1/2},$$

so all its eigenvalues are positive real numbers; if at least one other stiffness is positive definite, they lie strictly below 1.

Thus variational response is always monotone in its own Hessian geometry. Attractivity asks for the stronger Euclidean statement

$$h^T J_i h = h^T \operatorname{Sym}(K^{-1}K_i)h \geq 0. \tag{15}$$

Failure is therefore a non-normality or turning-angle phenomenon rather than a negative physical stiffness.

3.1 A local operator-angle certificate

The radial form is not needed for the basic matrix argument. Let $\phi(q) = \arccos(2\sqrt{q}/(q+1))$ denote the maximal turning angle of a positive definite matrix of condition number at most q.

Proposition 3.2 (Configuration-dependent stiffness certificate)**.** *At any regular variational configuration, the response to movement of p_i is attractive if*

$$\boxed{\phi(\kappa(K)) + \phi(\kappa(K_i)) \le \frac{\pi}{2}.} \tag{16}$$

This statement applies to arbitrary positive definite local stiffness tensors, not only to radial ones.

Proof. Put $A = K^{-1}$, $B = K_i$. The vectors Ah and Bh make angles at most $\phi(\kappa(K))$ and $\phi(\kappa(K_i))$, respectively, with h. Under (16) their mutual angle is at most $\pi/2$, and therefore

$$h^T K^{-1} K_i h = (Ah) \cdot (Bh) \ge 0.$$

$\square$

The proposition separates two levels of the theory. A universal force-law theorem must control the stiffnesses on every possible configuration. A particular symmetric configuration may be much better conditioned. In particular, if K is a scalar matrix, then J_i has positive symmetric part for every positive definite K_i, irrespective of its anisotropy.

4 Exact elasticity classification

For a symmetric positive definite operator T, write $\kappa(T)$ for its spectral condition number. The maximal turning-angle inequality gives

$$\frac{z^T T z}{\|z\|\,\|Tz\|} \ge \frac{2\sqrt{\kappa(T)}}{\kappa(T)+1} \tag{17}$$

for every nonzero z; equivalently the angle between z and Tz is at most

$$\phi(q) = \arccos\frac{2\sqrt{q}}{q+1} \qquad \text{if } \kappa(T) \le q. \tag{18}$$

This is the geometric form of the Kantorovich–Wielandt inequality and is closely related to the first antieigenvalue of a positive operator [10, 11].

Define

$$q_* = 3 + 2\sqrt{2}, \qquad q_*^{-1} = 3 - 2\sqrt{2}. \tag{19}$$

Lemma 4.1 (Elasticity-preserving collision regularization)**.** *Assume*

$$a_- := 3 - 2\sqrt{2} \le \eta(r) \le a_+ := 3 + 2\sqrt{2} \qquad (r > 0).$$

For every $\varepsilon > 0$ there exists a strictly convex potential $\psi_\varepsilon \in C^2([0,\infty))$ with force $F_\varepsilon = \psi_\varepsilon'$ such that:

(i) $F_\varepsilon(r) = F(r)$ *for* $r \ge \varepsilon$*;*

(ii) $F_\varepsilon(r) = c_\varepsilon r$ *for all sufficiently small* r*;*

(iii) $a_- \le rF_\varepsilon'(r)/F_\varepsilon(r) \le a_+$ *for* $r > 0$*;*

(iv) $\psi_\varepsilon \to \psi$ *uniformly on every compact radius interval, up to irrelevant additive constants.*

Proof. Put $u = \log r$ and $g(u) = \log F(e^u)$. Then $g'(u) = \eta(e^u) \in [a_-, a_+]$. Choose a smooth cutoff χ which is zero far to the left of $\log \varepsilon$, equals one at and to the right of $\log \varepsilon$, and takes values in $[0, 1]$. On $u \le \log \varepsilon$ prescribe

$$g_\varepsilon'(u) = 1 + \chi(u)\big(g'(u) - 1\big)$$

and choose the integration constant so that $g_\varepsilon(\log \varepsilon) = g(\log \varepsilon)$; for $u \ge \log \varepsilon$ put $g_\varepsilon = g$. Since $1 \in [a_-, a_+]$, the derivative remains in the same interval. Far to the left $g_\varepsilon'(u) = 1$, so $F_\varepsilon(r) = \exp(g_\varepsilon(\log r)) = c_\varepsilon r$ near zero. Thus F_ε extends C^1 to the origin and its integral gives a strictly convex C^2 potential there.

The lower elasticity bound implies $F(r) \to 0$ as $r \downarrow 0$. Both F and F_ε are increasing on $[0, \varepsilon]$ and are bounded there by $F(\varepsilon)$. Hence, after fixing additive constants,

$$\sup_{0 \le r \le R} |\psi_\varepsilon(r) - \psi(r)| \le 2\varepsilon F(\varepsilon) \longrightarrow 0$$

for every fixed R, proving compact-uniform convergence. □

Theorem 4.2 (Exact radial attractivity classification)**.** *Let ψ satisfy the standing assumptions. The following are equivalent:*

(i) for every finite Euclidean configuration, including collisions, moving one input from p_i to $p_i + h$ satisfies the global finite displacement law

$$h \cdot \big[X_\psi(P_{i,h}) - X_\psi(P)\big] \ge 0;$$

(ii) X_ψ is infinitesimally attractive at every regular finite configuration, for every number of inputs and in every dimension;

(iii) it is infinitesimally attractive at every regular finite planar configuration;

(iv)

$$\boxed{3 - 2\sqrt{2} \le \eta(r) \le 3 + 2\sqrt{2} \qquad \textit{for every } r > 0.} \tag{20}$$

Thus (20) is the exact pointwise criterion for the full finite displacement law in the radial variational location class.

Proof. The implication (i)⇒(ii) follows by differentiating the finite inequality at a regular configuration, and (ii)⇒(iii) is immediate.

For (iv)⇒(ii), put

$$q = \max\left\{\sup_r \eta(r), \frac{1}{\inf_r \eta(r)}\right\} \le 3 + 2\sqrt{2}.$$

Formula (10) gives $\kappa(K_i) \le q$. If m_i is the smallest eigenvalue of K_i, then

$$m_i I \preceq K_i \preceq q m_i I,$$

and summation gives $\kappa(K) \le q$. Set $A = K^{-1}$, $B = K_i$. Both Ah and Bh turn a nonzero vector h by at most

$$\phi(q) = \arccos \frac{2\sqrt{q}}{q + 1}.$$

Since $2\phi(q) \le \pi/2$ exactly for $q \le 3 + 2\sqrt{2}$,

$$h^T K^{-1} K_i h = (Ah) \cdot (Bh) \ge 0.$$

Thus every regular response is attractive.

To prove (iii)⇒(iv), fix any radius $r_0 > 0$, write $k = \eta(r_0)$ and $a_0 = F(r_0)/r_0$, and set

$$d = (1,0)^T, \qquad e = 2^{-1/2}(1,1)^T,$$

$$D = I + (k-1)dd^T = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = I + (k-1)ee^T = \frac{1}{2}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}.$$

Place m antipodal pairs at radius r_0 on the d-axis and one antipodal pair on the e-axis. Pairwise force cancellation makes the origin the unique minimizer and

$$K = 2a_0(mD + B).$$

For movement of one anchor in the e-pair,

$$J_e = \frac{1}{2}(mD + B)^{-1}B. \tag{21}$$

As $m \to \infty$,

$$m \operatorname{Sym} J_e \longrightarrow \frac{1}{2}\operatorname{Sym}(D^{-1}B) = \frac{1}{2}H,$$

where

$$\det H = -\frac{(k+1)^2(k^2 - 6k + 1)}{16k^2}.$$

If $k \notin [3-2\sqrt{2}, 3+2\sqrt{2}]$, then H is indefinite, so a sufficiently large finite planar configuration has a negative response direction. Hence the elasticity band is necessary at every radius. Repeated anchors may be replaced by arbitrarily close distinct antipodal pairs without changing the sign.

Finally assume (iv) and prove the global finite law (i). Apply Theorem 4.1. Every ψ_ε is C^2 at collisions, has positive-definite total Hessian everywhere, and obeys the same elasticity band. Its location map is therefore C^1 along any finite single-anchor motion, and the already proved infinitesimal inequality integrates along the entire segment to

$$h \cdot \left[X_{\psi_\varepsilon}(P_{i,h}) - X_{\psi_\varepsilon}(P)\right] \geq 0.$$

For a fixed motion segment all anchors and minimizers stay in one compact set. Compact-uniform convergence of the potentials gives uniform convergence of the corresponding finite energy functions there; uniqueness of the minimizer then implies $X_{\psi_\varepsilon} \to X_\psi$ at both endpoints. Passing to the limit proves the finite displacement law for X_ψ, including configurations with collisions. □

Remark 4.3 (Strictness)**.** If the two elasticity inequalities are strict at every radius, then every *fixed finite regular configuration* has positive-definite symmetric responses: only finitely many radii occur, so their condition numbers have a strict maximum below the critical value. A uniform gap from the two elasticity endpoints is needed only for a quantitative positivity bound that is uniform over all configurations.

4.1 A quantitative fixed-radius obstruction

The preceding necessity argument can be made finite and explicit. For the configuration used in the proof, a direct calculation from (21) gives

$$\det \operatorname{Sym} J_e = -\frac{P_m(k)}{16\big(k^2m + 2km^2 + 2km + 2k + m\big)^2}, \tag{22}$$

where

$$P_m(k) = (k+1)^2\Big[(k^2 - 6k + 1)m^2 - 8km\Big] - 16k^2. \tag{23}$$

When $k^2 - 6k + 1 > 0$, this is positive for all sufficiently large m, so (22) is negative. One explicit sufficient condition is

$$m > \frac{4k\big((k+1) + \sqrt{2}\,|k-1|\big)}{(k+1)(k^2 - 6k + 1)}. \tag{24}$$

Thus a single out-of-band value of η at one radius produces a finite planar counterexample using only anchors on the circle of that radius.

Corollary 4.4 (Global power-growth characterization)**.** *Under the standing assumptions, X_ψ satisfies the global finite displacement law if and only if, for every $0 < r < s$,*

$$\boxed{\left(\frac{s}{r}\right)^{3-2\sqrt{2}} \le \frac{F(s)}{F(r)} \le \left(\frac{s}{r}\right)^{3+2\sqrt{2}}.} \tag{25}$$

Proof. Integrating $\eta(t) = \mathrm{d}\log F(t)/\,\mathrm{d}\log t$ from r to s shows that (20) implies (25). Conversely, divide the logarithm of (25) by $\log(s/r)$ and let $s \to r$ to recover the pointwise elasticity bounds. ☐

Corollary 4.5 (No bounded-influence force)**.** *A radial force whose location functional satisfies the global finite displacement law cannot saturate at infinity. More precisely, for every fixed $r_0 > 0$,*

$$F(r_0)\left(\frac{r}{r_0}\right)^{3-2\sqrt{2}} \le F(r) \le F(r_0)\left(\frac{r}{r_0}\right)^{3+2\sqrt{2}} \qquad (r \ge r_0).$$

In particular $F(r) \to \infty$. Hence every smooth strictly convex robust force with $\eta(r) \to 0$ at large radius fails the global finite displacement law.

Corollary 4.6 (Heterogeneous radial response systems)**.** *Consider*

$$\Phi(x; P) = \sum_i \psi_i(\|x - p_i\|),$$

where every ψ_i is increasing and strictly convex and every elasticity

$$\eta_i(r) = \frac{r\psi_i''(r)}{\psi_i'(r)}$$

lies in $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$. Then the minimizing rule has attractive regular infinitesimal response, and the finite displacement law holds along any collision-free regular path.

Proof. Each local stiffness has condition number at most q_*, irrespective of its scalar prefactor. Their sum has condition number at most q_*, so the turning-angle proof applies unchanged; integration gives the regular-path statement. ☐

Remark 4.7 (Labels and invariance)**.** If the functions ψ_i are genuinely different and attached to fixed indices, this is a theorem about a labelled mechanical response system, not a permutation-invariant center or location functional. Permutation equivariance is restored only if interaction types are permuted together with their anchors. Similarity equivariance is still exceptional and is governed by Theorem 2.2 in the identical-interaction case.

Figure 2: The logarithmic elasticity $\eta = rF'/F$ is the ratio of radial to tangential stiffness. The displayed band is the exact pointwise criterion for the global finite displacement law in every number of points and every dimension.

4.2 Positive mixtures of admissible powers

The exact elasticity classification is not restricted to pure powers.

Corollary 4.8 (Exact criterion for finite positive mixtures of powers)**.** *Let*

$$\psi(r) = \sum_{j=1}^{m} c_j \frac{r^{p_j}}{p_j}, \qquad c_j > 0, \qquad p_j > 1. \tag{26}$$

Then X_ψ satisfies the global finite displacement law if and only if

$$\boxed{4 - 2\sqrt{2} \le p_j \le 4 + 2\sqrt{2} \qquad \textit{for every } j.} \tag{27}$$

The sufficient direction also holds for positive integral mixtures supported in the same exponent interval whenever differentiation under the integral is legitimate.

Proof. Here

$$F(r) = \sum_j c_j r^{p_j - 1},$$

and

$$\eta(r) = \frac{\sum_j c_j (p_j - 1) r^{p_j - 1}}{\sum_j c_j r^{p_j - 1}}. \tag{28}$$

Thus $\eta(r)$ is a positive weighted average of the numbers $p_j - 1$, so (27) implies the elasticity band. Conversely, after combining equal exponents, as $r \downarrow 0$ the smallest exponent dominates and as $r \to \infty$ the largest exponent dominates:

$$\eta(r) \to p_{\min} - 1 \quad (r \downarrow 0), \qquad \eta(r) \to p_{\max} - 1 \quad (r \to \infty).$$

The exact elasticity classification therefore forces both extremal exponents, and hence every exponent, to lie in the stated interval. $\square$

For example, every positive combination of the quadratic, cubic, and quartic costs produces an attractive radial location functional. Except when all exponents coincide, such a fixed mixture is not scale equivariant by Theorem 2.2 and therefore does not belong to the Heart class of Papers I and II. The theorem supplies an infinite-dimensional cone in the broader radial location-functional class; the pointwise elasticity band, rather than representation as a mixture, is the fundamental object.

5 Power-distance centers: the exact absolute interval

For

$$\psi_p(r) = \frac{r^p}{p}, \qquad p > 1, \tag{29}$$

the variational center is

$$M_p(P) = \arg\min_x \sum_i \|x - p_i\|^p. \tag{30}$$

Its force law is $F(r) = r^{p-1}$, so

$$\eta(r) = p - 1. \tag{31}$$

By Theorem 2.2, these are precisely the radial variational rules that belong to the original similarity-invariant center class. The exact elasticity classification gives the sharp interval, and the stronger result below shows that outside it failure is witnessed by a noncollinear triangle: three points suffice.

Theorem 5.1 (Exact attractivity interval for power centers)**.** *For $p > 1$, the following are equivalent:*

(i) M_p is attractive for every finite Euclidean configuration in every dimension;

(ii) M_p is attractive for every noncollinear three-point configuration in $\mathbb{R}^2$;

(iii)

$$\boxed{4 - 2\sqrt{2} \le p \le 4 + 2\sqrt{2}.} \tag{32}$$

Thus this is the exact maximal interval in which the power laws are attractive centers in the original Papers I–II sense, and failure outside it already occurs for triangles.

The implication (iii)⇒(i), including collisions and finite motions, is the power-law specialization of Theorem 4.2. The implication (i)⇒(ii) is trivial. We prove (ii)⇒(iii) by constructing asymptotically extremal triangles.

Put

$$k = p - 1 > 0, \qquad \alpha = \frac{k-1}{k}. \tag{33}$$

At a minimizer placed at the origin, define the *force vector*

$$f_i = r_i^k e_i. \tag{34}$$

The equilibrium equation is simply

$$\sum_i f_i = 0. \tag{35}$$

Conversely, any nonzero force vector f is realized by

$$u = \|f\|^{1/k-1} f, \qquad p = -u, \tag{36}$$

because $\|u\|^k u/\|u\| = f$. Hence any finite collection of nonzero force vectors summing to zero gives a stationary configuration with center at the origin; strict convexity of the power objective makes that stationary point the unique minimizer. The corresponding stiffness is

$$K(f) = \|f\|^\alpha B(e), \qquad B(e) = I + (k-1)ee^T, \qquad e = f/\|f\|. \tag{37}$$

Let

$$d = (1, 0)^T, \qquad e = \frac{1}{\sqrt{2}}(1, 1)^T, \tag{38}$$

and put

$$D = B(d) = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = B(e) = \frac{1}{2}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}. \tag{39}$$

A direct calculation gives

$$H := \mathrm{Sym}(D^{-1}B) = \begin{pmatrix} \dfrac{k+1}{2k} & \dfrac{k^2-1}{4k} \\ \dfrac{k^2-1}{4k} & \dfrac{k+1}{2} \end{pmatrix} \tag{40}$$

and

$$\boxed{\det H = -\frac{(k+1)^2(k^2-6k+1)}{16k^2}.} \tag{41}$$

The trace is positive. Therefore H has one negative eigenvalue exactly when

$$k < 3 - 2\sqrt{2} \qquad \text{or} \qquad k > 3 + 2\sqrt{2}. \tag{42}$$

5.1 The upper failure: $k > 3 + 2\sqrt{2}$

Assume $k > 1$, so $\alpha > 0$. For $\delta > 0$, choose three force vectors

$$f_1 = d - \frac{\delta}{2}e, \qquad f_2 = -d - \frac{\delta}{2}e, \qquad f_3 = \delta e. \tag{43}$$

They sum to zero. Use (36) to obtain three anchors whose M_p center is exactly the origin. They are noncollinear for all sufficiently small $\delta > 0$: the first two anchors have positive second coordinate and opposite first-coordinate signs, whereas the third anchor is $-\delta^{1/k}e$, with negative second coordinate and first coordinate tending to zero. Hence, for small δ, the third anchor lies below the segment joining the first two. As $\delta \downarrow 0$,

$$K_1 + K_2 \longrightarrow 2D, \qquad K_3 = \delta^\alpha B.$$

For movement of the third point,

$$S_3 = \mathrm{Sym}\big((K_1 + K_2 + K_3)^{-1}K_3\big),$$

and hence

$$\delta^{-\alpha}S_3 \longrightarrow \frac{1}{2}H. \tag{44}$$

If $k > 3 + 2\sqrt{2}$, (41) is negative. Therefore S_3 has a negative eigenvalue for all sufficiently small positive δ. This gives an explicit family of nondegenerate triangle counterexamples.

5.2 The lower failure: $0 < k < 3 - 2\sqrt{2}$

Now $0 < k < 1$, so $\alpha < 0$. Choose

$$f_1 = e, \qquad f_2 = -e - \delta d, \qquad f_3 = \delta d. \tag{45}$$

Again $\sum_i f_i = 0$, so the force-space realization produces anchors with $M_p = 0$. They are noncollinear for every $\delta > 0$: the first anchor is $-e$, the second lies on the ray through $e + \delta d$, so their joining line crosses the d-axis at a positive coordinate, whereas the third anchor is $-\delta^{1/k}d$ on the negative d-axis. Here

$$K_1 = B, \qquad K_2 \longrightarrow B, \qquad K_3 = \delta^\alpha D.$$

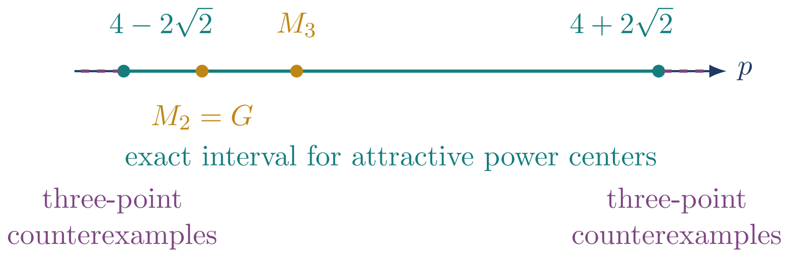

Figure 3: For the similarity-invariant power-distance centers the global finite attractivity interval is exact. Outside it, nearly degenerate planar three-point configurations realize the negative turning-angle mechanism.

Since $\alpha < 0$, the third stiffness dominates as $\delta \downarrow 0$. For movement of the first point,

$$\delta^\alpha S_1 = \delta^\alpha \,\mathrm{Sym}\big((K_1 + K_2 + K_3)^{-1} K_1\big) \longrightarrow H. \tag{46}$$

Multiplication by the positive scalar δ^α does not change the sign of eigenvalues. If $k < 3 - 2\sqrt{2}$, H is indefinite, and so is S_1 for sufficiently small δ.

Together the two constructions prove (ii)⇒(iii) and complete the proof of Theorem 5.1.

For orientation, taking finite rather than asymptotic values of δ already produces visible violations. For the force-space families above, one finds for example

p	branch	δ	smallest eigenvalue of the indicated S_i
7	upper, moved point 3	10^{-4}	-4.75×10^{-6}
1.15	lower, moved point 1	0.2	-1.09×10^{-5}

These numbers are not part of the proof; they simply show that the exact asymptotic construction produces ordinary finite counterexamples without a separate numerical search.

Remark 5.2 (Why the sharp constant appears)**.** The two counterexample families make the total stiffness and the moved-point stiffness approach two matrices with the same condition number but principal axes separated by 45°. Equation (41) is precisely the boundary at which their combined turning can reach 90°. Thus the constant $3 + 2\sqrt{2}$ is not an artifact of a loose condition-number estimate; it is geometrically attained.

6 Special configurations and limiting centers

Global attractivity is a worst-case property. A center can be attractive on a symmetry stratum for parameter values that fail globally. Paper II showed, for example, that on every regular-base axial simplex the complete family M_p, $1 < p < \infty$, is attractive when one restricts the allowed deformation to that one-parameter symmetry family [3]. This does not contradict Theorem 5.1: the counterexamples necessarily leave the symmetry stratum.

At a configuration for which the total stiffness is a scalar matrix, every individual response is symmetric positive. Regular polygons, regular simplices, and other isotropic configurations therefore have locally attractive M_p response for every $p > 1$. Again, local isotropy must not be confused with absolute attractivity.

As $p \downarrow 1$, M_p tends to the spatial median; as $p \to \infty$, it tends to the minimum-enclosing-ball center. Both limiting constructions are known to fail SC-monotonicity in general [9]. In a triangle the latter agrees with the circumcenter on the acute branch, where failure is already visible from a single Jacobian. With

$$A = (-1, 0), \qquad B = (1, 0), \qquad C = (q, r), \quad r > 0,$$

the circumcenter is

$$O(q,r) = \left(0, \frac{q^2 + r^2 - 1}{2r}\right),$$

and its symmetric response to motion of C is

$$S_C = \begin{pmatrix} 0 & q/(2r) \\ q/(2r) & (r^2 - q^2 + 1)/(2r^2) \end{pmatrix}, \qquad \det S_C = -\frac{q^2}{4r^2} < 0 \tag{47}$$

whenever $q \neq 0$.

7 Continuum radial locations and the large-N limit

The radial-energy framework passes naturally from finite clouds to mass distributions. Let μ be a compactly supported probability measure on $\mathbb{R}^d$, and define

$$\Phi_\psi(x;\mu) = \int \psi(\|x-y\|)\,\mathrm{d}\mu(y), \qquad X_\psi(\mu) = \arg\min_x \Phi_\psi(x;\mu). \tag{48}$$

Under the standing increasing strict-convexity assumptions the minimizer is unique.

Let $v \in L^\infty(\mu;\mathbb{R}^d)$, put

$$T_\varepsilon(y) = y + \varepsilon v(y), \qquad \boxed{\mu_\varepsilon = (T_\varepsilon)_\# \mu},$$

and write $x = X_\psi(\mu)$. For a nonzero displacement vector u, define the radial stiffness kernel

$$\mathscr{K}(u) = \frac{F(\|u\|)}{\|u\|}\left[I + (\eta(\|u\|) - 1)\frac{uu^T}{\|u\|^2}\right].$$

For differentiation we use either of two sufficient hypotheses. In the smooth-kernel case, the force map $g(u) = F(\|u\|)u/\|u\|$ extends to a C^1 map at $u = 0$, with a continuous stiffness kernel $\mathscr{K} = Dg$ on the relevant compact displacement set. Set $g(0) = 0$ and use this extension in all integrals. This case includes $\psi = r^p/p$ for $p \geq 2$ and allows positive density near the center. In the separated-support case we impose the following hypothesis. There exist a neighborhood U of x, $\varepsilon_0 > 0$, and $G \in L^1(\mu)$ such that for μ-almost every y and all $z \in U$, $|\varepsilon| \leq \varepsilon_0$, the vector $z - T_\varepsilon(y)$ is nonzero and the corresponding radial stiffness satisfies

$$\|\mathscr{K}(z - T_\varepsilon(y))\|\,(1 + \|v(y)\|) \leq G(y). \tag{49}$$

In either case assume also that the total stiffness at x is positive definite. The separated-support condition excludes positive mass in the whole neighbourhood U at $\varepsilon = 0$; it is not a theorem for arbitrary uniformly filled solids with an interior center. Condition (49) is a concrete dominated-differentiation hypothesis; for example it holds automatically if the relevant support stays a positive distance from U and ψ is C^2 on the resulting compact radius interval.

Put

$$r(y) = \|x-y\|, \qquad e(y) = \frac{x-y}{r(y)},$$

and, for $r(y) > 0$,

$$K_y = \mathscr{K}(x-y) = \frac{F(r(y))}{r(y)}\left[I + (\eta(r(y)) - 1)e(y)e(y)^T\right], \qquad K = \int K_y\,\mathrm{d}\mu(y). \tag{50}$$

At $r(y) = 0$, the smooth-kernel branch uses its continuous extension in place of the displayed quotient.

Theorem 7.1 (Continuum response)**.** *Under either sufficient hypothesis above and positive-definite total stiffness, the first variation of $X_\psi(\mu_\varepsilon)$ at $\varepsilon = 0$ is*

$$\boxed{\dot{x} = K^{-1} \int K_y v(y) \, \mathrm{d}\mu(y).} \tag{51}$$

If η satisfies (20), then

$$\mathrm{Sym}(K^{-1} K_y) \succeq 0$$

for μ-almost every y. Consequently, if a measurable portion E of the material is displaced infinitesimally and rigidly by h, then

$$h \cdot \dot{x} = \int_E h^T K^{-1} K_y h \, \mathrm{d}\mu(y) \geq 0. \tag{52}$$

Proof. In the smooth-kernel case, compactness and bounded v give an integrable uniform derivative bound; in the separated-support case use the stated domination. The implicit function theorem applies because the total stiffness is positive definite. Differentiate the continuum equilibrium equation under the integral. The variation in x contributes $K\dot{x}$, while movement of the material contributes $-\int K_y v(y) \, \mathrm{d}\mu(y)$ to the force balance, giving (51). Under the elasticity hypothesis, for almost every y the tensor K_y is either zero (at a smooth zero-stiffness collision) or positive definite with condition number at most q_*. If $m(y) = \lambda_{\min}(K_y)$, then

$$m(y) I \preceq K_y \preceq q_* m(y) I.$$

Integration yields

$$\left(\int m \, \mathrm{d}\mu \right) I \preceq K \preceq q_* \left(\int m \, \mathrm{d}\mu \right) I,$$

so $\kappa(K) \leq q_*$. The turning-angle proof of Theorem 4.2 therefore applies pointwise to $K^{-1} K_y$. Equation (52) follows by taking $v = h \mathbf{1}_E$. □

Remark 7.2 (Atoms, material subsets, and null sets)**.** Moving an atom of mass $m > 0$ contributes the corresponding response weighted by m. Moving a measurable subset E of positive μ-measure gives the integral in (52). By contrast, moving one individual point of a non-atomic measure changes the measure on a null set and has no first-order effect. These are distinct continuum notions and should not be conflated.

Corollary 7.3 (Finite material displacement along a regular path)**.** *Translate a measurable portion E by th, $0 \leq t \leq 1$, and suppose the continuum configuration remains regular in the preceding sense for the entire path. Under (20),*

$$\boxed{h \cdot \left[X_\psi(\mu_1) - X_\psi(\mu_0) \right] \geq 0.}$$

Proof. Apply Theorem 7.1 at each t and integrate $h \cdot \dot{x}(t) \geq 0$ from 0 to 1. □

This is the continuum analogue of moving one atom or a positive-mass material subset. It provides a centerwise large-N bridge for each fixed radial law; it does not by itself establish convergence of Hearts or Chests, which would require estimates uniform over a class of rules.

Corollary 7.4 (Exact radial criterion for atomic measures)**.** *Within the standing radial class, the following are equivalent:*

(i) every compactly supported finite atomic probability measure has attractive response under motion of any one atom, at every regular configuration;

(ii) the elasticity satisfies (20) for every radius.

Proof. Sufficiency is the atomic case of Theorem 7.1. Necessity follows from the fixed-radius finite configurations in the necessity proof of Theorem 4.2, viewed as equal-mass atomic measures. $\square$

Corollary 7.5 (Exact continuum power threshold with atomic measures)**.** *If compactly supported atomic probability measures are admitted, then the power center is attractive under arbitrary motion of one atom for every such measure if and only if*

$$4 - 2\sqrt{2} \le p \le 4 + 2\sqrt{2}.$$

Proof. Sufficiency follows from Theorem 7.1 together with the same smooth regularization used in Theorem 5.1; positive atom masses merely multiply the corresponding local stiffnesses and do not affect the condition-number argument. Necessity already follows from the equal-weight three-atom measures corresponding to the triangle counterexamples in that theorem. $\square$

Proposition 7.6 (Stability under compact empirical approximation)**.** *Assume the standing increasing strict-convexity hypotheses on ψ. Let μ_n and μ be probability measures supported in one fixed compact set, with $\mu_n \rightharpoonup \mu$. Then*

$$X_\psi(\mu_n) \longrightarrow X_\psi(\mu). \tag{53}$$

In particular the conclusion holds for empirical measures of uniformly filling point clouds, including the stronger W_∞ convergence used in Paper I.

Proof. All minimizers lie in the convex hull K of the common compact support. The jointly continuous function

$$g(x, y) = \psi(\|x - y\|)$$

is uniformly continuous on $K \times K$. Given $\varepsilon > 0$, choose a finite net $x_1, \ldots, x_m$ in K such that every function $g(x, \cdot)$, $x \in K$, is uniformly within ε of one of the $g(x_j, \cdot)$. Weak convergence gives convergence of the integrals for these finitely many test functions. Hence

$$\sup_{x \in K} \left| \int g(x, y)\, \mathrm{d}\mu_n(y) - \int g(x, y)\, \mathrm{d}\mu(y) \right| \longrightarrow 0.$$

Thus the energies converge uniformly on K. Every accumulation point of minimizers minimizes the limiting energy, and strict convexity gives the unique minimizer $X_\psi(\mu)$. $\square$

Theorem 7.7 (Stability of the power-generated inner region)**.** *Let $Q \subset \mathbb{R}^d$ be compact and let μ_n, μ be probability measures supported in Q, with $\mu_n \rightharpoonup \mu$. Put $I = [4 - 2\sqrt{2}, 4 + 2\sqrt{2}]$ and*

$$K_{\mathrm{pow}}(\nu) = \mathrm{conv}\{M_p(\nu) : p \in I\}.$$

Then these are nonempty compact convex sets and

$$\sup_{p \in I} \|M_p(\mu_n) - M_p(\mu)\| \longrightarrow 0, \qquad d_H(K_{\mathrm{pow}}(\mu_n), K_{\mathrm{pow}}(\mu)) \longrightarrow 0.$$

This is a region theorem for the convex family generated by the admissible power laws, not for the full class of all attractive center rules.

Proof. All minimizers lie in the compact set $C = \mathrm{conv}\, Q$. The function $(p, x, y) \mapsto \|x - y\|^p$ is jointly continuous on $I \times C \times Q$, including $x = y$, since $\min I > 1$. A finite-net argument as in the preceding proposition gives uniform convergence of the energies over $(p, x) \in I \times C$. Every fixed exponent has a unique minimizer. Joint continuity of minimizers in (ν, p) follows by taking convergent subsequences of minimizers and passing the minimization inequality to the limit. If uniform convergence in p failed, a sequence p_n with a convergent subsequence would contradict this joint continuity. The generating curves are compact, so their convex hulls are compact in finite dimension. Pairing the same exponents in any convex combination bounds the Hausdorff distance of the two hulls by the displayed supremum. $\square$

Remark 7.8. Each power rule is replication invariant. For finite atomic measures, positive weights multiply the stiffness tensors and preserve the attraction proof. For compactly supported material measures, finite translation of a selected portion also obeys the response inequality: approximate the selected and unselected submeasures separately by finite atomic measures, apply sequential parallel atom displacements, and pass to the limit by centerwise consistency. This finite-law argument does not require differentiating a singular kernel. For a centrally symmetric measure, every power center and hence this entire region is the symmetry center.

8 A contrasting physical family: skeletal mass centers

A center can have an entirely natural mass interpretation and still fail attractivity. Let

$$S = [p_0, \dots, p_n]$$

be a nondegenerate n-simplex and distribute one common density per unit m-volume over the complete m-skeleton. If $\mathcal{F}_m(S)$ is its set of m-faces,

$$V_F = \mathrm{Vol}_m(F), \qquad c_F = \frac{1}{m+1} \sum_{p_j \in F} p_j, \qquad W_m = \sum_{F \in \mathcal{F}_m(S)} V_F,$$

then

$$C_{n,m}(S) = \frac{1}{W_m} \sum_{F \in \mathcal{F}_m(S)} V_F c_F. \tag{54}$$

The two extremes are always the ordinary centroid:

$$C_{n,0} = C_{n,n} = \frac{1}{n+1} \sum_{i=0}^{n} p_i.$$

Intermediate skeleta are different because moving a vertex changes both the face centroids and the face masses.

For a fixed moved vertex p_i, put

$$g_{F,i} = \nabla_{p_i} V_F.$$

Differentiation gives

$$D_{p_i} C_{n,m} = \lambda_i I + \frac{1}{W_m} \sum_{\substack{F \in \mathcal{F}_m(S) \\ p_i \in F}} (c_F - C_{n,m}) g_{F,i}^T, \tag{55}$$

where

$$\lambda_i = \frac{\sum_{F \ni p_i} V_F}{(m+1) W_m}.$$

The first term is positive, but the second is a redistribution term with no general sign.

8.1 The triangle exception

For a triangle, the only intermediate skeleton is the boundary wire. With usual side lengths a, b, c and perimeter $P = a + b + c$,

$$C_{2,1} = \frac{(b+c)A + (c+a)B + (a+b)C}{2P} = X(10). \tag{56}$$

Paper IV proves the boundary-wire center's global attractivity and sharp symmetric-Jacobian spectrum, and derives the incenter relation $X(1) = 3G - 2X(10)$ within the same affine family [2]. The new phenomenon here is the failure of intermediate skeletal constructions in dimension three.

8.2 Failure for tetrahedra

The favorable triangle behavior already fails in dimension three. Consider

$$A = (0,0,0), \quad B = (1,0,0), \quad C = (0,1,0), \quad D = \left(3, -2, \frac{1}{2}\right). \tag{57}$$

For uniform mass on the six edges, movement of B gives

$$S_B^{(1)} \approx \begin{pmatrix} 0.05383645 & 0.09272015 & -0.01702402 \\ 0.09272015 & 0.11182305 & 0.00911974 \\ -0.01702402 & 0.00911974 & 0.18542614 \end{pmatrix},$$

with eigenvalues approximately

$$-0.01613152, \qquad 0.17907040, \qquad 0.18814676.$$

For uniform mass on the four triangular faces,

$$S_B^{(2)} \approx \begin{pmatrix} 0.18591654 & 0.05476205 & -0.11401076 \\ 0.05476205 & 0.04313936 & 0.02130216 \\ -0.11401076 & 0.02130216 & 0.16660395 \end{pmatrix},$$

with eigenvalues

$$-0.00242750, \qquad 0.10459154, \qquad 0.29349581.$$

Thus neither $C_{3,1}$ nor $C_{3,2}$ is absolutely attractive. The tetrahedron has rational vertices but its edge and face masses involve radicals. The supplementary Mathematica file supplied with this paper computes the exact algebraic Jacobians and verifies negative exact determinants of the two symmetric response matrices, giving algebraic certificates rather than merely floating-point evidence.

The lesson is important: being the center of positive physical mass is not a sufficient mechanism for attractivity. Radial equilibrium centers are controlled instead by the operator geometry of stiffness transfer.

9 Physical interpretation and examples

Let the radial force magnitude be $F(r)$. Then

$$k_t(r) = \frac{F(r)}{r}, \qquad k_r(r) = F'(r), \qquad \eta(r) = \frac{k_r(r)}{k_t(r)}.$$

The exact classification says that a star of identical radial elements satisfies the global finite displacement law for every finite anchor configuration if and only if

$$3 - 2\sqrt{2} \le \frac{k_r(r)}{k_t(r)} \le 3 + 2\sqrt{2} \tag{58}$$

at every extension used by the system.

potential / force law	elasticity η	conclusion from this paper
$\psi = r^2/2,\ F = r$	1	centroid; attractive center
$\psi = r^3/3,\ F = r^2$	2	cubic power center; strict regular response
$\psi = r^4/4,\ F = r^3$	3	attractive power center
$\psi = r^p/p$	$p - 1$	attractive center iff $4 - 2\sqrt{2} \le p \le 4 + 2\sqrt{2}$
positive mixture of different admissible powers	weighted average of $p_j - 1$	attractive radial location functional, generally not scale-equivariant
smooth saturating robust force	$\eta \to 0$	fails the global finite displacement law

The exact power theorem also gives a mechanical interpretation of the sharp constant. Outside the interval, one can arrange three anchors so that the total stiffness is strongly anisotropic in one pair of axes while the moved anchor has nearly the same anisotropy rotated by 45°. Each stiffness is positive, yet the Euclidean work $h \cdot \delta x$ becomes negative in one direction.

10 Relation to aggregation, location, statistics, and operator theory

Pérez-Fernández, De Baets, and Gagolewski developed a taxonomy of multidimensional alternatives to ordinary order monotonicity, including the SC-monotonicity condition used here [9]. Their work classifies a range of standard aggregation rules and, in particular, records failures of SC-monotonicity for familiar Euclidean location procedures. The new contribution of the present paper is different in kind: within the radial variational class it gives an exact mechanical characterization of SC-monotonicity by the radial-to-tangential stiffness ratio, proves pointwise necessity, and for the scale-free power subclass gives three-point planar witnesses outside the sharp interval.

The response identity

$$D_{p_i} X_\psi = K^{-1} K_i$$

is an instance of implicit differentiation of a parameterized strictly convex minimizer. Differentiation through Fréchet means in substantially more general geometric settings is developed by Lou et al. [12]; the present Euclidean calculation is elementary but exploits the resulting Hessian-inverse structure for a monotonicity question rather than for gradient propagation. In robust multivariate statistics, radial estimating equations and their influence functions are classical; Maronna's theory of multivariate M-estimators provides a nearby statistical context [13]. The incompatibility between bounded influence and our exact lower elasticity bound gives a particularly direct robustness–attractivity trade-off.

For probability measures, consistency of Fréchet-type sample locations has a large literature. Bhattacharya and Patrangenaru, for example, prove consistency and asymptotic results for intrinsic and extrinsic sample means on manifolds [14]. Our Theorem 7.6 is a simpler Euclidean compact-support statement for one fixed radial potential and is not intended as a replacement for that theory.

Finally, the matrix inequality underlying the sharp constant belongs to the geometry of positive noncommuting operator products. Gustafson's operator angle and antieigenvalue formalism identifies

$$\frac{2\sqrt{\kappa}}{\kappa + 1}$$

as the cosine of the maximal turning angle of a positive definite operator of condition number κ [10, 11]. His companion work on positive noncommuting operator products is especially close to the product $K^{-1} K_i$ appearing here [15]. The fixed-radius and power counterexamples show that this operator-angle threshold is geometrically sharp in the location problem.

11 Open problems

1. **Minimal witnesses outside the elasticity band.** The fixed-radius proof gives a finite planar counterexample whenever $\eta(r_0)$ leaves the sharp band, but the required number of equal-radius anchors can grow as $\eta(r_0)$ approaches an endpoint. Determine the minimal number of distinct unweighted points needed as a function of the out-of-band elasticity. For constant elasticity, the power-law construction shows that three points already suffice.

2. **Boundary laws and semidefinite response.** Classify radial force laws that touch one or both sharp elasticity endpoints on nontrivial radius sets. Determine when the resulting

global response has genuine null directions and when summation of differently oriented local stiffnesses makes the response strictly positive despite pointwise boundary elasticity.

3. **Sharp heterogeneous criteria.** The uniform elasticity band already allows different positive radial potentials at different anchors. Determine the optimal nonuniform criterion in terms of the actual collection of local stiffness condition numbers and their orientations.

4. **Anisotropic energies.** Proposition 3.2 applies to full positive stiffness tensors at one configuration. Develop invariant global classes of anisotropic potentials for which the operator-angle condition can be certified everywhere.

5. **Continuum attainable sets.** For a bounded region with uniform density, study the set of radial location values generated by admissible force laws. Theorem 7.7 settles the convex power-generated region on common compact support. Extend this to larger coherent classes with uniform continuity estimates, and to unbounded measures under explicit tail assumptions.

6. **Skeletal mass centers.** Classify all pairs (n, m) for which $C_{n,m}$ is absolutely attractive. Is the triangle edge center the only nontrivial intermediate skeletal exception?

7. **Dynamics.** If the central node is given mass and damping, study whether static attractivity has a dynamical counterpart: does a displaced anchor produce nonnegative integrated response or work before the system returns to equilibrium?

8. **Statistical robustness versus attractivity.** Quantify the trade-off between a force law whose influence saturates at large radius and the Euclidean monotonic response required by attractivity.

12 Conclusion

Radial variational location functionals reveal a clean physical mechanism behind attractivity. The response of an equilibrium location is a partition of total stiffness,

$$J_i = K^{-1}K_i,$$

and is automatically positive in the Hessian metric. Euclidean attractivity asks whether the local and global stiffnesses can turn a vector so far apart that their dot product becomes negative. The logarithmic elasticity $\eta = rF'/F$ controls exactly this anisotropy.

The sharp band

$$3 - 2\sqrt{2} \le \eta(r) \le 3 + 2\sqrt{2} \qquad (r > 0)$$

therefore converts the finite displacement law into an exact mechanical classification: an identical radial interaction satisfies the global finite law for every finite configuration if and only if its radial-to-tangential stiffness ratio stays in this band at every radius. A violation at even one radius is detected by a finite planar configuration lying entirely on the corresponding circle. The broader radial class is generally not scale equivariant; Theorem 2.2 shows that the original invariant-center axiom selects exactly the pure power laws. For that subclass, three points already suffice outside the interval:

$$\boxed{M_p \text{ is an attractive center} \iff 4 - 2\sqrt{2} \le p \le 4 + 2\sqrt{2}.}$$

The failure outside this interval already occurs for three planar points and is caused by nearly maximal relative turning of two positive stiffness operators.

The continuum theorem shows, under explicit integrability and regularity hypotheses, that the same response law survives the passage from finite point clouds to distributed matter; along a regular material-motion path the infinitesimal inequality integrates to the finite displacement law. The skeletal examples show, conversely, that not every physically motivated center is attractive. The central lesson is therefore structural rather than metaphorical: physical centrality becomes mathematically useful when its linear response can be expressed by positive local operators whose relative geometry is controlled.

A Algebra of the sharp two-matrix limit

For completeness, with

$$D = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix},$$

one has

$$D^{-1}B = \begin{pmatrix} \dfrac{k+1}{2k} & \dfrac{k-1}{2k} \\ \dfrac{k-1}{2} & \dfrac{k+1}{2} \end{pmatrix}.$$

Therefore

$$\operatorname{Sym}(D^{-1}B) = \begin{pmatrix} \dfrac{k+1}{2k} & \dfrac{k^2-1}{4k} \\ \dfrac{k^2-1}{4k} & \dfrac{k+1}{2} \end{pmatrix}.$$

Its determinant is

$$\begin{aligned}\det \operatorname{Sym}(D^{-1}B) &= \frac{(k+1)^2}{4k} - \frac{(k^2-1)^2}{16k^2} \\ &= \frac{(k+1)^2}{16k^2}\big(4k-(k-1)^2\big) \\ &= -\frac{(k+1)^2(k^2-6k+1)}{16k^2}.\end{aligned}$$

The determinant vanishes at

$$k = 3 \pm 2\sqrt{2},$$

which are reciprocal. This reciprocity corresponds mechanically to interchanging a radial stiffness excess with a tangential stiffness excess.

B Reproducible algebraic certificates

The supplementary Wolfram Language file `Attractive_Centers_III_certificates.wl` reproduces the finite computations used in the paper using exact input data. It contains routines for skeletal simplex centers, symbolic vertex Jacobians, the two tetrahedral examples of Eq. (57), and the finite power-law witnesses at $p = 7, \delta = 10^{-4}$ and $p = 23/20, \delta = 1/5$. For the skeletal examples it verifies exactly that

$$\det S_B^{(1)} < 0, \qquad \det S_B^{(2)} < 0,$$

where the determinants are algebraic numbers because the edge lengths and face areas contain radicals. For the two power examples it likewise verifies an exact negative determinant of the relevant planar symmetric response. The script reconstructs the corresponding triangle vertices from the exact force vectors, verifies exact force balance and noncollinearity, and prints high-precision numerical values for comparison with the tables in the text. It is supplied as an exact symbolic verification route. A second, standard-library Python verifier encloses the determinants using rational interval arithmetic and exact integer root bounds; its completed transcript is included. All four certified determinant intervals have strictly negative upper endpoints. Thus the published numerical examples have independently checkable sign certificates without relying on finite differences or a commercial algebra system. The simpler NumPy script is retained as a numerical cross-check, with assertions and a nonzero exit on failure. The Wolfram script's run transcript is not claimed unless generated by a successful local run.

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