Dihedral-Angle Sums in Orthocentric, Power-k, and Isodynamic Tetrahedra

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Abstract

For a nondegenerate Euclidean tetrahedron T, let $\Sigma(T)$ be the sum of its six internal dihedral angles. We prove that the full orthocentric family has the exact range $[6\arccos(1/3), 3\pi)$, with the regular tetrahedron as the unique minimizer; its positive additive branch has the smaller range $[6\arccos(1/3), 5\pi/2)$. The proof combines a constant polar-measure sum with a strictly convex lower envelope for solid-angle content. For Power-k metrics $d_{ij}^k = y_i + y_j$, with positive pair sums and Euclidean realizability, we prove that the exact range is $(2\pi, 3\pi)$ whenever $k < 1$, $k \neq 0$, or $k > 2$. For the two-pair subfamily $y = (u, u, v, v)$, we establish the regular lower bound throughout $1 \leq k \leq 2$, strict monotonicity in the squared-edge exponent for unequal parameters, and the exact range $[6\arccos(1/3), 2\pi + 2\arcsin(2^{1/k-1}))$. The general lower bound for $1 < k < 2$ remains conjectural: we prove strict local minimality at the regular metric and give a feasible exponent deformation as a sufficient route to the global inequality. Finally, we determine the isodynamic range $(2\pi, 3\pi)$ and record a separate isogonic reduction problem in an appendix.

Keywords. Power-k tetrahedron; orthocentric tetrahedron; 4-ball tetrahedron; balloon tetrahedron; isodynamic tetrahedron; isogonic tetrahedron; dihedral angle; solid angle; polar sine; kite.
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1 Introduction

For a nondegenerate Euclidean tetrahedron T, write $\Sigma(T)$ for the sum of its six internal dihedral angles. The sharp universal bounds

$$2\pi < \Sigma(T) < 3\pi \tag{1}$$

were proved by Jarden (Juzuk) in 1941; see [5] and the later short treatment of Gaddum [3]. Neither endpoint is attained by a nondegenerate tetrahedron.

The regular tetrahedron has six equal dihedral angles $\arccos(1/3)$ and therefore

$$\Sigma_{\text{reg}} = 6\arccos\frac{1}{3} = 2.350959312\ldots\ \pi. \tag{2}$$

The purpose of this article is to determine when this regular value remains the lower endpoint after one restricts the metric to natural tetrahedral families, and when the infimum falls all the way to the universal value 2π.

We use the terminology *Power-k tetrahedron* for the family described by

$$d_{ij}^k = y_i + y_j, \qquad k \in \mathbb{R} \setminus \{0\}, \tag{3}$$

subject to positive pair sums and Euclidean realizability. This notation exposes two important members of the same family. At $k = 2$ one obtains the signed additive metric

$d_{ij}^2 = y_i + y_j$, which is exactly the orthocentric family [2, 4]. At $k = 1$ the metric is edge-additive. The latter family is the classical 4-ball or balloon family; its equivalent characterizations are discussed, for example, by Katsuura [7]. Korotov and Křížek state the corresponding regular lower bound for the balloon family [8, Theorem 9]. Our results are independent of that bound; its role is discussed in Remark 6.6.

Our main results and reductions are as follows.

(i) We prove the regular lower bound for the full orthocentric, hence Power-2, family, including the component with one negative additive parameter. Equilateral-base kites then show that the exact range is $[6\arccos(1/3), 3\pi)$.

(ii) For the positive Power-2 branch we combine the new lower bound with the sharp nonobtuse upper bound to obtain the exact range $[6\arccos(1/3), 5\pi/2)$.

(iii) For every $k < 1$, $k \neq 0$, and every $k > 2$, we construct convex-planar degenerations inside the full Power-k family and prove that the exact range is the full universal interval $(2\pi, 3\pi)$. The interval $1 < k < 2$ remains open. We prove strict local minimality of the regular tetrahedron throughout this interval and give a sufficient exponent-monotonicity criterion using Schoenberg's snowflake theorem. For the complete two-pair subfamily we prove both the lower bound and exponent monotonicity, and obtain its exact range (Theorem 7.1).

(iv) In the normalized positive power-mean parametrization, the limit $k \to 0$ is the isodynamic metric. We determine its exact range to be $(2\pi, 3\pi)$ by two explicit endpoint paths and continuity.

The orthocentric lower bound is the conceptual core of the paper. Hajja and Hammoudeh proved that the four polar measures of the corner angles of every orthocentric tetrahedron add to π [4, Theorem 1(b)]. We combine that identity with an elementary comparison between polar measure and ordinary solid-angle content. The minimizing acute corner at fixed polar measure is equiangular, and its content is a strictly convex function of the polar measure. Jensen's inequality then gives the regular lower bound in the all-acute case, while a separate estimate handles the possible right or obtuse corner.

The following table summarizes the status proved or used in this paper. The entry at $k = 0$ is a normalized limiting family, not a literal substitution in (3).

Parameter/family	Range of Σ	Status
$k < 1$, $k \neq 0$, full $\mathcal{P}_k$	$(2\pi, 3\pi)$	Proved
$k = 1$, full $\mathcal{P}_1$	$[\Sigma_{\mathrm{reg}}, 3\pi)$	Stated in [8]
$1 < k < 2$, full $\mathcal{P}_k$	$[\Sigma_{\mathrm{reg}}, 3\pi)$	Conjectured
$1 \leq k \leq 2$, two-pair	$[\Sigma_{\mathrm{reg}}, U_k)$	Proved here
$k = 2$, full $\mathcal{P}_2$	$[\Sigma_{\mathrm{reg}}, 3\pi)$	Proved
$k = 2$, positive $\mathcal{P}_2^+$	$[\Sigma_{\mathrm{reg}}, 5\pi/2)$	Proved
$k > 2$, full $\mathcal{P}_k$	$(2\pi, 3\pi)$	Proved
Normalized $k \to 0$: isodynamic	$(2\pi, 3\pi)$	Proved

Here $\Sigma_{\mathrm{reg}} = 6\arccos(1/3)$ and $U_k = 2\pi + 2\arcsin(2^{1/k-1})$. The two-pair row refers only to $y = (u, u, v, v)$ and does not assert the range of the full family.

The organization is as follows. Section 2 introduces the two angle measures, the Power-k terminology and its special cases, and the constant polar-sum theorem. Section 3 proves the local comparison and its crucial convexity. Section 4 proves the orthocentric lower bound, treating the all-acute and non-acute cases separately. Section 5 uses kites to obtain the exact orthocentric range and the range of the positive Power-2 branch. Section 6 proves the exterior Power-k ranges and gives rigorous partial results for $1 < k < 2$. Section 7 settles the complete two-pair subfamily. Section 8 proves the exact isodynamic range. The discussion puts the different proof mechanisms in perspective.

2 Preliminaries

2.1 Content and the dihedral sum

Consider a trihedral angle with vertex at the origin and unit vectors u_1, u_2, u_3 along its three arms. Set

$$a = u_2 \cdot u_3, \qquad b = u_3 \cdot u_1, \qquad c = u_1 \cdot u_2.$$

Thus a, b, c are the cosines of its three planar subangles. Its polar sine is

$$P = \sqrt{1 - a^2 - b^2 - c^2 + 2abc} > 0. \tag{4}$$

The ordinary solid-angle content $\Omega \in (0, 2\pi)$ is the area of the spherical triangle cut out by the three arms on the unit sphere. The standard determinant formula is

$$\Omega = 2\,\mathrm{atan2}\big(P, 1 + a + b + c\big), \tag{5}$$

where $\mathrm{atan2}(y, x) \in (0, \pi)$ denotes the argument of $x + iy$ when $y > 0$. This notation keeps the correct branch when $1 + a + b + c$ is negative.

Let $\Omega_1, \ldots, \Omega_4$ be the contents at the four vertices of a tetrahedron, and let $\theta_1, \ldots, \theta_6$ be its internal dihedral angles. At each vertex, spherical excess gives

$$\Omega_i = (\text{sum of the three incident dihedral angles}) - \pi.$$

Summing over the four vertices counts each dihedral angle twice. Hence

$$\sum_{i=1}^{4} \Omega_i = 2\Sigma(T) - 4\pi, \qquad \Sigma(T) = 2\pi + \frac{1}{2}\sum_{i=1}^{4} \Omega_i. \tag{6}$$

2.2 Pure corners and polar measure

A corner is called *acute*, *right*, or *obtuse* when its three planar subangles are respectively all acute, all right, or all obtuse. These are the pure corners relevant here. Following [4], define their polar measure by

$$\beta = \begin{cases} \arcsin P, & \text{for an acute corner,} \\ \pi/2, & \text{for a right corner,} \\ \pi - \arcsin P, & \text{for an obtuse corner.} \end{cases} \tag{7}$$

In every case $\sin \beta = P$.

An orthocentric tetrahedron admits real additive parameters $a_1, \ldots, a_4$ such that

$$d_{ij}^2 = a_i + a_j \qquad (i \neq j), \tag{8}$$

and all pair sums are positive; see [2, 4]. Consequently at most one a_i can be nonpositive. The cosine rule shows that all three planar subangles at vertex i have the sign of a_i. Thus every corner is pure, at least three corners are acute, and at most one is right or obtuse.

Definition 2.1 (Power-k tetrahedron)**.** Let $k \in \mathbb{R} \setminus \{0\}$. A nondegenerate Euclidean tetrahedron is a *Power-k tetrahedron* if its edge lengths satisfy

$$d_{ij}^k = y_i + y_j > 0 \qquad (i \neq j) \tag{9}$$

for real parameters $y_1, \dots, y_4$. We denote the family by $\mathcal{P}_k$. Its *positive branch* $\mathcal{P}_k^+$ consists of the members admitting $y_i > 0$ for all i.

The requirement that all pair sums in (9) be positive already implies that at most one y_i is negative. Multiplying every y_i by the same positive constant changes only the scale of the tetrahedron, so the Power-k family has three shape parameters.

Proposition 2.2 (Intrinsic Power-k relation)**.** *For a nondegenerate Euclidean tetrahedron with edge lengths d_{ij} and a fixed $k \neq 0$, membership in the Power-k family is equivalent to*

$$d_{12}^k + d_{34}^k = d_{13}^k + d_{24}^k = d_{14}^k + d_{23}^k. \tag{10}$$

Thus the definition is intrinsic and does not depend on a chosen set of parameters.

Proof. If (9) holds, each expression in (10) equals $y_1 + y_2 + y_3 + y_4$. Conversely, put $A_{ij} = d_{ij}^k$ and assume the three opposite-pair sums are equal. Define

$$y_1 = \frac{A_{12} + A_{13} - A_{23}}{2}, \quad y_2 = \frac{A_{12} + A_{23} - A_{13}}{2}, \quad y_3 = \frac{A_{13} + A_{23} - A_{12}}{2},$$

and $y_4 = A_{14} - y_1$. Then $y_i + y_j = A_{ij}$ for the three edges among $1, 2, 3$. The two remaining equalities follow from $A_{14} + A_{23} = A_{13} + A_{24} = A_{12} + A_{34}$, proving (9) for every edge. $\square$

The special exponents 1 and 2 deserve to be separated carefully. For $k = 2$, (9) is exactly (8); hence

$$\mathcal{P}_2 = \{\text{orthocentric tetrahedra}\}. \tag{11}$$

The positive branch $\mathcal{P}_2^+$ is the interior of the nonobtuse component of the orthocentric family; allowing zero parameters gives its right-corner boundary. Indeed, writing $y_i = x_i^2$ gives

$$d_{ij}^2 = x_i^2 + x_j^2, \qquad x_i > 0, \tag{12}$$

whereas an orthocentric tetrahedron with one pure obtuse corner requires one negative additive parameter.

For $k = 1$, the face inequalities remove the signed component automatically. If $d_{ij} = y_i + y_j$, then in any face containing vertex i,

$$d_{ij} + d_{ik} > d_{jk} \quad \iff \quad 2y_i > 0.$$

Hence every realizable Power-1 tetrahedron has $y_i > 0$ for all i. Thus $\mathcal{P}_1 = \mathcal{P}_1^+$ and this family is exactly the classical edge-additive, balloon, or 4-ball family; see [7] for these equivalent viewpoints. The logical status of its proposed sharp range is discussed in Section 6.

Finally, the positive branch has a natural normalized formulation. For $x_i > 0$ set

$$L_{ij}^{(k)} = \left(\frac{x_i^k + x_j^k}{2} \right)^{1/k}. \tag{13}$$

For each fixed $k \neq 0$ this parametrizes the same shapes as $\mathcal{P}_k^+$. If instead the x_i are held fixed while $k \to 0$, then

$$\lim_{k \to 0} L_{ij}^{(k)} = \sqrt{x_i x_j}. \tag{14}$$

Writing $p_i = \sqrt{x_i}$ gives $d_{ij} = p_i p_j$, the isodynamic metric. Thus the isodynamic family is the normalized positive $k \to 0$ limit, not a literal $k = 0$ instance of (9).

We use the following established result as an external input.

Theorem 2.3 (Constant polar-measure sum)**.** *For every nondegenerate orthocentric tetrahedron, the polar measures of its four corner angles satisfy*

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = \pi. \tag{15}$$

Proof. This is [4, Theorem 1(b), Section 5]. The branch choice in (7) is essential in the obtuse case. □

3 The least content of an acute corner

The key local problem is the following: if the polar measure of an acute corner is fixed, how small can its ordinary content be? The minimizing corner is equiangular.

Define

$$\phi(q) = (1-q)^2(1+2q) = 1 - 3q^2 + 2q^3, \qquad 0 \le q \le 1. \tag{16}$$

The function ϕ decreases strictly from 1 to 0. Given $\beta \in [0, \pi/2]$, let $q = q(\beta) \in [0,1]$ be the unique solution of

$$\sin^2\beta = \phi(q). \tag{17}$$

Define

$$f(\beta) = 3\arccos\frac{q(\beta)}{1+q(\beta)} - \pi. \tag{18}$$

Geometrically, $f(\beta)$ is the content of the equiangular trihedral angle whose three planar subangles have cosine $q(\beta)$.

Lemma 3.1 (Acute-corner comparison)**.** *If an acute trihedral angle has polar measure β and content Ω, then*

$$\Omega \ge f(\beta). \tag{19}$$

Equality holds if and only if its three planar subangles are equal.

Proof. Use the notation $a, b, c > 0$ from (4), and put $s = a + b + c$. The elementary inequalities

$$a^2 + b^2 + c^2 \ge \frac{s^2}{3}, \qquad abc \le \left(\frac{s}{3}\right)^3$$

give

$$\begin{aligned} P^2 &= 1 - a^2 - b^2 - c^2 + 2abc \\ &\le 1 - \frac{s^2}{3} + \frac{2s^3}{27} = \phi\left(\frac{s}{3}\right). \end{aligned}$$

By (17), $P^2 = \phi(q(\beta))$. Since ϕ is decreasing,

$$q(\beta) \ge \frac{s}{3}, \qquad 1 + s \le 1 + 3q(\beta).$$

For fixed $P > 0$, the expression $2\,\mathrm{atan2}(P, X)$ decreases as X increases. Formula (5) therefore yields

$$\Omega \ge 2\arctan\frac{\sin\beta}{1+3q(\beta)}. \tag{20}$$

For an equiangular corner with $a = b = c = q$, the spherical cosine rule shows that each angle δ of the corresponding equilateral spherical triangle satisfies

$$\cos\delta = \frac{q}{1+q}.$$

Its spherical excess is $3\delta - \pi$, so the right-hand side of (20) is precisely $f(\beta)$. Equality in the two elementary inequalities above occurs simultaneously exactly when $a = b = c$. $\square$

Lemma 3.2 (Convexity)**.** *The function f is strictly increasing and strictly convex on $[0, \pi/2]$. Moreover,*

$$f(0) = 0, \qquad f'_+(0) = \frac{1}{2}, \qquad f(\beta) \geq \frac{\beta}{2}. \tag{21}$$

Proof. It is convenient to regard both β and f as functions of $q \in [0,1]$:

$$\beta(q) = \arcsin\big((1-q)\sqrt{1+2q}\big), \qquad f(q) = 3\arccos\frac{q}{1+q} - \pi.$$

For $0 < q < 1$, direct differentiation gives

$$\frac{d\beta}{dq} = -\frac{3}{\sqrt{(1+2q)(3-2q)}}, \qquad \frac{df}{dq} = -\frac{3}{(1+q)\sqrt{1+2q}}. \tag{22}$$

Consequently,

$$f'(\beta) = \frac{\sqrt{3-2q}}{1+q} > 0. \tag{23}$$

Differentiating once more and using (22),

$$f''(\beta) = \frac{(4-q)\sqrt{1+2q}}{3(1+q)^2} > 0. \tag{24}$$

Thus f is strictly increasing and strictly convex. As $\beta \downarrow 0$ we have $q \uparrow 1$, and (23) gives $f'_+(0) = 1/2$. The graph of a convex function lies above every supporting tangent, which proves $f(\beta) \geq \beta/2$. $\square$

At the regular value $\beta = \pi/4$, equation (17) gives $q = 1/2$, and hence

$$f\left(\frac{\pi}{4}\right) = 3\arccos\frac{1}{3} - \pi. \tag{25}$$

4 The lower bound

We first treat orthocentric tetrahedra whose four corners are acute.

Proposition 4.1 (All-acute case)**.** *If all four corners of an orthocentric tetrahedron T are acute, then*

$$\Sigma(T) \geq 6\arccos\frac{1}{3}. \tag{26}$$

Equality holds if and only if T is regular.

Proof. By Lemma 3.1, Theorem 2.3, Jensen's inequality, and (25),

$$\begin{aligned}\sum_{i=1}^{4} \Omega_i &\geq \sum_{i=1}^{4} f(\beta_i) \\ &\geq 4f\left(\frac{\beta_1+\beta_2+\beta_3+\beta_4}{4}\right) \\ &= 4f\left(\frac{\pi}{4}\right) \\ &= 12\arccos\frac{1}{3} - 4\pi.\end{aligned}$$

Equation (6) now gives (26).

Suppose equality holds. Strict convexity forces $\beta_1 = \cdots = \beta_4 = \pi/4$, and equality in Lemma 3.1 forces each corner to be equiangular. In particular, at any vertex the three planar subangles have cosine $1/2$. Using (8) at, say, vertex 4, their cosines are

$$\frac{a_4}{\sqrt{(a_4+a_i)(a_4+a_j)}} \qquad (1 \leq i < j \leq 3).$$

Their equality implies $a_1 = a_2 = a_3$, and their common value $1/2$ then implies $a_4 = a_1$. Thus all a_i are equal and all six edges are equal. Conversely, the regular tetrahedron plainly gives equality. □

It remains to ensure that the possible right or obtuse corner cannot lower the sum. In fact it leads to a stronger estimate.

Lemma 4.2 (Obtuse-corner comparison)**.** *If a pure obtuse corner has polar measure $\beta \in (\pi/2, \pi)$ and content Ω, then*

$$\Omega \geq \beta. \tag{27}$$

The same inequality holds with equality for a right corner, where $\Omega = \beta = \pi/2$.

Proof. For an obtuse corner write

$$a = -A, \qquad b = -B, \qquad c = -C, \qquad 0 < A, B, C < 1,$$

and set $r = A + B + C$. By (4),

$$1 - P^2 = A^2 + B^2 + C^2 + 2ABC.$$

Since $AB + AC + BC \geq ABC$,

$$r^2 = A^2 + B^2 + C^2 + 2(AB + AC + BC) \geq 1 - P^2.$$

Because $\beta = \pi - \arcsin P$, this says

$$r \geq \sqrt{1 - P^2} = -\cos\beta.$$

Thus $1 - r \leq 1 + \cos\beta$. Using the monotonicity of $\operatorname{atan2}(P, X)$ in its second argument,

$$\begin{aligned}\Omega &= 2\operatorname{atan2}(P, 1-r) \\ &\geq 2\operatorname{atan2}(\sin\beta, 1+\cos\beta) = \beta.\end{aligned}$$

For a right corner $a = b = c = 0$, and the assertion follows immediately from (5) and (7). □

Proposition 4.3 (Right and obtuse cases)**.** *If an orthocentric tetrahedron has a right or obtuse corner, then*

$$\Sigma(T) \geq \frac{19\pi}{8} > 6 \arccos \frac{1}{3}. \tag{28}$$

Proof. Let the exceptional corner be number 4. Then $\beta_4 \geq \pi/2$, while the other three corners are acute. Lemmas 3.2 and 4.2, together with Theorem 2.3, give

$$\begin{aligned} \sum_{i=1}^{4} \Omega_i &\geq \beta_4 + \frac{1}{2}(\beta_1 + \beta_2 + \beta_3) \\ &= \beta_4 + \frac{1}{2}(\pi - \beta_4) = \frac{\pi + \beta_4}{2} \geq \frac{3\pi}{4}. \end{aligned}$$

Equation (6) yields $\Sigma(T) \geq 19\pi/8$.

For completeness, $19\pi/8 > 6 \arccos(1/3)$ is equivalent to $\cos(19\pi/48) < 1/3$. The latter follows from

$$\cos^2 \frac{19\pi}{48} = \frac{1 - \cos(5\pi/24)}{2} < \frac{1}{9}.$$

Indeed,

$$\cos^2 \frac{5\pi}{24} = \frac{1}{2}\left(1 + \frac{\sqrt{6} - \sqrt{2}}{4}\right).$$

Since $\sqrt{3} < 7/4$, we have $(\sqrt{6} - \sqrt{2})^2 = 8 - 4\sqrt{3} > 1$. Hence

$$\cos^2 \frac{5\pi}{24} > \frac{5}{8} > \frac{49}{81},$$

which proves $\cos(5\pi/24) > 7/9$. □

Combining the two cases proves the central theorem.

Theorem 4.4 (Sharp lower bound)**.** *Every nondegenerate orthocentric tetrahedron satisfies*

$$\boxed{\Sigma(T) \geq 6 \arccos \frac{1}{3}}. \tag{29}$$

Equality holds if and only if T is regular.

5 Kites and the exact range

An equilateral-base kite has an equilateral base and three equal lateral edges. Normalize the base edge to 1 and denote the lateral edge by t. The nondegeneracy condition is

$$t > \frac{1}{\sqrt{3}}. \tag{30}$$

Every such kite is orthocentric. In fact, in (8) take the three base parameters equal to $1/2$ and the apex parameter equal to $t^2 - 1/2$.

The three base-edge dihedral angles and the three lateral-edge dihedral angles give the sum

$$K(t) = 3\left(\arccos \frac{1}{\sqrt{3(4t^2 - 1)}} + \arccos \frac{2t^2 - 1}{4t^2 - 1}\right). \tag{31}$$

This formula follows directly by placing the apex above the center of the equilateral base and taking scalar products of face normals.

Differentiation simplifies completely:

$$K'(t) = \frac{6(t-1)}{(4t^2-1)\sqrt{3t^2-1}}. \tag{32}$$

Thus K decreases on $(1/\sqrt{3}, 1]$, reaches its unique minimum at the regular tetrahedron $t = 1$, and increases for $t > 1$. Moreover,

$$K(1) = 6\arccos\frac{1}{3}, \qquad \lim_{t\downarrow 1/\sqrt{3}} K(t) = 3\pi. \tag{33}$$

Consequently the short-kite branch alone realizes every value in $[6\arccos(1/3), 3\pi)$.

Theorem 5.1 (Exact range)**.** *The set of dihedral-angle sums of nondegenerate orthocentric tetrahedra is*

$$\boxed{\{\Sigma(T) : T \text{ orthocentric}\} = \left[6\arccos\frac{1}{3},\, 3\pi\right).} \tag{34}$$

Proof. Theorem 4.4 gives the left endpoint and its equality case. The universal upper bound in (1) gives $\Sigma(T) < 3\pi$. Finally, (31)–(33) show that the orthocentric kite subfamily realizes every value between these bounds. □

The distinction between signed and positive parameters can now be quantified exactly. In the kite parametrization above, the signed apex parameter is $a_4 = t^2 - 1/2$. The flattening branch $t \downarrow 1/\sqrt{3}$ therefore enters the component $a_4 < 0$ once $t < 1/\sqrt{2}$. It belongs to the full orthocentric family but not to the positive Power-2 branch $\mathcal{P}_2^+$.

Corollary 5.2 (Exact range for the positive Power-2 branch)**.** *For tetrahedra in $\mathcal{P}_2^+$,*

$$\boxed{\left\{\Sigma(T) : T \in \mathcal{P}_2^+\right\} = \left[6\arccos\frac{1}{3},\, \frac{5\pi}{2}\right).} \tag{35}$$

Proof. Every tetrahedron in $\mathcal{P}_2^+$ is orthocentric, so Theorem 4.4 supplies the lower bound and its equality case. Its dihedral angles are nonobtuse; equivalently, this also follows from the positive formula (75) below. The sharp general bound for a nonobtuse tetrahedron is

$$\Sigma(T) < \frac{5\pi}{2};$$

see [1, Theorem 3.1, $n = 3$].

It remains only to realize every intermediate value. Take the kite branch $t \geq 1$. It lies in $\mathcal{P}_2^+$: take the three base parameters $y_1 = y_2 = y_3 = 1/2$ and the apex parameter

$$y_4 = t^2 - \frac{1}{2} > 0.$$

By (32), $K(t)$ increases strictly from $K(1) = 6\arccos(1/3)$, and formula (31) gives

$$\lim_{t\to\infty} K(t) = 3\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \frac{5\pi}{2}.$$

Thus this one-parameter positive Power-2 subfamily realizes the entire interval in (35). □

6 The Power-k family

We now vary the exponent in the Power-k family $\mathcal{P}_k$ of Definition 2.1. The signed parameters y_i are kept directly; for even or nonintegral k, a negative y_i need not be represented as a real kth power. We first record two simple geometric facts used at both ends of the range.

Lemma 6.1 (Kites occur for every exponent)**.** *For every $k \neq 0$, $\mathcal{P}_k$ contains every equilateral-base kite.*

Proof. Take $y_1 = y_2 = y_3 = u > 0$ and $y_4 = v$. The base edge and the three lateral edges are

$$b = (2u)^{1/k}, \qquad \ell = (u+v)^{1/k}.$$

Given an arbitrary ratio $t = \ell/b > 0$, put

$$\frac{v}{u} = 2t^k - 1. \tag{36}$$

Then $v > -u$, so all pair sums are positive and at most v is negative. For $t > 1/\sqrt{3}$ the resulting kite is nondegenerate. Hence all kites in Section 5 occur. ☐

Lemma 6.2 (Convex planar degeneration)**.** *Suppose a continuous family of nondegenerate tetrahedra tends to four distinct coplanar points in convex position. Then the dihedral angles at the two diagonals of the limiting quadrilateral tend to π, the other four dihedral angles tend to 0, and consequently*

$$\Sigma(T) \longrightarrow 2\pi. \tag{37}$$

Proof. Orient the limiting plane horizontally. Along a boundary edge, the two incident faces approach the plane from opposite normal orientations, so the internal dihedral angle tends to 0. Along a diagonal, the third vertices lie on opposite sides of the diagonal in the limiting plane; the two face normals therefore approach the same orientation and the internal dihedral angle tends to π. There are four boundary edges and two diagonals. Equivalently, the statement follows by taking scalar products of the face normals before passing to the limit. ☐

6.1 The exact range for $k < 1$, $k \neq 0$

Theorem 6.3. *For every $k < 1$, $k \neq 0$, the dihedral-angle sums of Power-k tetrahedra have the exact range*

$$\boxed{\{\Sigma(T) : T \in \mathcal{P}_k\} = (2\pi, 3\pi).} \tag{38}$$

The lower endpoint is already approached with all four parameters positive.

Proof. The universal bounds (1) give containment in $(2\pi, 3\pi)$. Lemma 6.1 and the short-kite branch give every value in $[6\arccos(1/3), 3\pi)$.

It remains to reach the lower endpoint. Put

$$(y_1, y_2, y_3, y_4) = (r, r, 1, 1), \qquad r > 0, \tag{39}$$

and set $q = 2/k$. The two distinguished opposite edges and the four equal cross-edges are

$$a = (2r)^{1/k}, \qquad c = 2^{1/k}, \qquad \ell = (1+r)^{1/k}.$$

They are realized by

$$
\begin{aligned}
A &= (-a/2, 0, 0), & B &= (a/2, 0, 0), \\
C &= (0, h, c/2), & D &= (0, h, -c/2),
\end{aligned}
\qquad h = \frac{1}{2}\sqrt{F_k(r)}, \tag{40}
$$

where

$$
F_k(r) = 4(1+r)^q - (2r)^q - 2^q. \tag{41}
$$

At $r = 1$ all six edges are equal and $F_k(1) > 0$. If $0 < k < 1$, then $q > 2$ and

$$
F_k(0) = 4 - 2^q < 0.
$$

If $k < 0$, then $q < 0$ and $F_k(r) \to -\infty$ as $r \downarrow 0$. In either case, the component of $\{r : F_k(r) > 0\}$ containing $r = 1$ has a lower endpoint $r_0 \in (0,1)$ with $F_k(r_0) = 0$.

As $r \downarrow r_0$, the height in (40) tends to zero. The segments AB and CD remain nonzero, meet orthogonally at their common midpoint, and are the two diagonals of the limiting convex quadrilateral. Lemma 6.2 therefore gives

$$
\Sigma(r) \longrightarrow 2\pi, \qquad \Sigma(1) = 6 \arccos \frac{1}{3}.
$$

The sum is continuous on $(r_0, 1]$. The intermediate value theorem supplies every value in $(2\pi, 6\arccos(1/3)]$; monotonicity is unnecessary. Combining this path with the kite path proves (38). $\square$

6.2 The exact range for $k > 2$

Fix $k > 2$ and put $p = 1/k \in (0, 1/2)$. Consider the signed Power-k path

$$
(y_1, y_2, y_3, y_4) = (1, t, t, -rt), \qquad t > 0, \qquad 0 \le r < 1. \tag{42}
$$

Set

$$
L = (1+t)^p, \quad M = (1-rt)^p, \quad a = (2t)^p, \quad b = ((1-r)t)^p. \tag{43}
$$

The face on vertices $2, 3, 4$ is isosceles with sides a, b, b and is nondegenerate precisely when

$$
r < r_c := 1 - 2^{1-k}. \tag{44}
$$

For $r < r_c$ use the coordinates

$$
\begin{aligned}
B &= (-a/2, 0, 0), & C &= (a/2, 0, 0), \\
D &= (0, \eta, 0), & A &= (0, z, h),
\end{aligned} \tag{45}
$$

where

$$
\eta = \sqrt{b^2 - a^2/4}, \qquad z = \frac{L^2 - M^2 + b^2 - a^2/2}{2\eta}, \qquad h^2 = L^2 - a^2/4 - z^2. \tag{46}
$$

These formulas impose $AB = AC = L$ and $AD = M$.

Lemma 6.4 (Boundary layer of the signed path)**.** *Let $k > 2$ and use (42)–(46). For every sufficiently small $t > 0$ there is a first value $r_*(t) \in (0, r_c)$ such that*

$$
h^2 > 0 \quad (0 \le r < r_*(t)), \qquad h(r_*(t)) = 0.
$$

Moreover, if $c_0 = 1 - r_c = 2^{1-k}$*, then*

$$r_*(t) = r_c - (\lambda_* + o(1))t^{2p}, \qquad \lambda_* = \frac{2^{4p-6}}{2p\, c_0^{2p-1}} > 0. \tag{47}$$

At the limiting planar configuration one has $z < 0 < \eta$*, so the four points are in convex position. Consequently*

$$\Sigma(t,r) \longrightarrow 2\pi \quad \textit{as } r \uparrow r_*(t). \tag{48}$$

Proof. At $r = 0$,

$$\eta = t^p\sqrt{1 - 2^{2p-2}}, \qquad z = O(t^p), \qquad h^2 \longrightarrow 1 \quad (t \downarrow 0),$$

so the path begins in the nondegenerate region for small t.

Write $\delta = r_c - r$ and $c_0 = 1 - r_c$. Since $c_0^{2p} = 2^{2p-2}$, the first expression in (46) gives the exact identity

$$\eta^2 = t^{2p}\big((c_0 + \delta)^{2p} - c_0^{2p}\big). \tag{49}$$

Let

$$N(t,r) = L^2 - M^2 + b^2 - a^2/2$$

be the numerator of z. In the critical layer $\delta = \lambda t^{2p}$, with fixed $\lambda > 0$, Taylor expansion yields

$$\eta \sim t^{2p}\sqrt{2p\, c_0^{2p-1}\lambda}, \tag{50}$$

$$N(t,r) \sim -2^{2p-2}t^{2p}, \tag{51}$$

because $L^2 - M^2 = O(t) = o(t^{2p})$ when $2p < 1$. Therefore

$$z \longrightarrow -\frac{2^{2p-3}}{\sqrt{2p\, c_0^{2p-1}\lambda}}, \qquad h^2 \longrightarrow 1 - \frac{\lambda_*}{\lambda}. \tag{52}$$

In particular, the limiting sign of h^2 is positive for $\lambda > \lambda_*$ and negative for $0 < \lambda < \lambda_*$.

It remains to rule out an earlier loss of feasibility. Suppose $t_n \downarrow 0$, $r_n < r_c$, and put $\delta_n = r_c - r_n$. If $\delta_n/t_n^{2p} \to \infty$, then (49) and the expression for N give $z(t_n, r_n) \to 0$, hence $h^2(t_n, r_n) \to 1$. If instead δ_n/t_n^{2p} stays bounded away from zero and infinity, a subsequence lies in a critical layer and (52) applies. Consequently, for any fixed $\lambda_+ > \lambda_*$ and all sufficiently small t, $h^2(t,r) > 0$ throughout $0 \le r \le r_c - \lambda_+ t^{2p}$, whereas for any $0 < \lambda_- < \lambda_*$,

$$h^2(t, r_c - \lambda_- t^{2p}) < 0.$$

By continuity there is a first zero $r_*(t)$ between these two values. Letting $\lambda_\pm \to \lambda_*$ proves (47).

At this first zero, (52) shows $z < 0$ for small t, while $\eta > 0$ because $r_*(t) < r_c$. In the plane $h = 0$, the segment AD therefore crosses BC at an interior point, so A, B, D, C are four distinct points in convex position and AD, BC are their diagonals. Lemma 6.2 then proves (48). □

Theorem 6.5. *For every* $k > 2$*, the dihedral-angle sums of Power-*k *tetrahedra have the exact range*

$$\boxed{\{\Sigma(T) : T \in \mathcal{P}_k\} = (2\pi, 3\pi).} \tag{53}$$

Proof. The universal bounds (1) give containment in $(2\pi, 3\pi)$. At $r = 0$ the base triangle BCD shrinks as $t \downarrow 0$ while its shape remains fixed, whereas the three edges from A tend to 1. The three base-edge dihedral angles tend to $\pi/2$, and the three lateral-edge dihedral angles tend to the three angles of the base triangle. Hence

$$\Sigma(t, 0) \longrightarrow \frac{5\pi}{2} \quad (t \downarrow 0). \tag{54}$$

Choose t so small that $\Sigma(t, 0) > 6 \arccos(1/3)$. By Lemma 6.4, the same connected feasible path has $\Sigma(t, r) \to 2\pi$ as $r \uparrow r_*(t)$. Continuity therefore supplies every value from 2π through the regular value. Lemma 6.1 supplies every value from the regular value to 3π. The universal strict bounds complete the proof. □

6.3 The remaining interval $1 < k < 2$

Remark 6.6 (Status of the Power-1 case). By the discussion after Definition 2.1, $\mathcal{P}_1$ is exactly the 4-ball (balloon) family. Theorem 9 of [8] states that its exact range is $[6 \arccos(1/3), 3\pi)$. Its kite subfamily rigorously realizes that whole interval, so the only unresolved logical step is exclusion of values below the regular value. The cited proof treats an isosceles spherical triangle and says that the nonsymmetric case follows by a "tedious calculation," but the calculation or a certificate for the global maximization is not given. We do not use that lower bound in any proof below.

Theorems 6.3, 5.1, and 6.5 suggest the following conjecture.

Conjecture 6.7 (Power-k window)**.** *For every $1 < k < 2$, every Power-k tetrahedron satisfies*

$$\Sigma(T) \geq 6 \arccos \frac{1}{3}, \tag{55}$$

with equality only for the regular tetrahedron. Consequently its exact range is

$$\left[6 \arccos \frac{1}{3}, 3\pi\right).$$

It is convenient to put $q = 2/k$, so $1 < q < 2$, and to write the squared edges as

$$D_{ij} = d_{ij}^2 = (y_i + y_j)^q. \tag{56}$$

The following observation removes a potential feasibility obstruction from any attempt to deform the exponent down to the orthocentric value $q = 1$.

Proposition 6.8 (Downward feasibility)**.** *Fix real parameters $y_1, \ldots, y_4$ with $y_i + y_j > 0$ for $i \neq j$. If, for some $q_0 > 0$, the six numbers*

$$d_{ij}(q_0) = (y_i + y_j)^{q_0/2}$$

are the edge lengths of a nondegenerate Euclidean tetrahedron, then the same is true for $d_{ij}(q) = (y_i + y_j)^{q/2}$ for every $0 < q \leq q_0$.

Proof. For $0 < q < q_0$ put $\alpha = q/q_0 \in (0, 1)$. Then

$$d_{ij}(q) = d_{ij}(q_0)^\alpha.$$

Schoenberg's snowflake theorem says that a Euclidean metric raised to a power $\alpha \in (0, 1)$ is again Euclidean [9]. Its strict finite form says that for distinct points and nonzero coefficients c_i with $\sum_i c_i = 0$,

$$\sum_{i,j} c_i c_j d_{ij}(q_0)^{2\alpha} < 0.$$

Equivalently, the centered Gram matrix of the four new points is positive definite on the three-dimensional zero-sum subspace. Hence the four points are affinely independent and form a nondegenerate tetrahedron. The case $q = q_0$ is the hypothesis. ☐

Whenever the metric is feasible, let $\Sigma_y(q)$ denote its dihedral-angle sum. Proposition 6.8 makes the following inequality meaningful on the whole interval from a given q down to 1.

Conjecture 6.9 (Exponent monotonicity)**.** *For every fixed admissible parameter vector y, the function $q \mapsto \Sigma_y(q)$ is nondecreasing on every feasible subinterval of $[1, 2]$.*

Proposition 6.10 (Conditional reduction)**.** *Conjecture 6.9 implies the Power-k window Conjecture 6.7. It also supplies the missing lower-bound step in Remark 6.6.*

Proof. Let $1 \le k < 2$, put $q = 2/k \in (1, 2]$, and suppose that the corresponding Power-k metric is feasible. Proposition 6.8 shows that the same parameters at $q = 1$ define a nondegenerate orthocentric tetrahedron. Therefore Theorem 4.4 and the conjectured monotonicity give

$$\Sigma_y(q) \ge \Sigma_y(1) \ge 6 \arccos \frac{1}{3}.$$

If equality holds, the orthocentric equality case forces all four parameters to be equal, and hence the original tetrahedron is regular. Finally, Lemma 6.1 and the short-kite branch realize every value from the regular value up to 3π, while the universal upper bound is strict. ☐

The next proposition proves unconditionally that the proposed equality case is a strict local minimum throughout the entire interval.

Proposition 6.11 (Strict local minimality)**.** *Let $q > 0$. Subject to a normalization of scale, the regular point $y_1 = y_2 = y_3 = y_4$ is a strict local minimum of the dihedral sum. More precisely, if $\sum_i v_i = 0$, then*

$$\Sigma_q(1 + \varepsilon v_1, \ldots, 1 + \varepsilon v_4) = 6 \arccos \frac{1}{3} + \frac{q^2}{12\sqrt{2}} \varepsilon^2 \sum_{i=1}^{4} v_i^2 + O(\varepsilon^3). \tag{57}$$

Proof. Base the vertex Gram matrix at vertex 1:

$$G_{ab} = \frac{D_{1,a+1} + D_{1,b+1} - D_{a+1,b+1}}{2}, \qquad 1 \le a, b \le 3,$$

where $D_{ii} = 0$, and put $H = G^{-1}$ and $\mathbf{e} = (1, 1, 1)^T$. The six dihedral cosines are

$$\frac{(H\mathbf{e})_a}{\sqrt{H_{aa}\mathbf{e}^T H\mathbf{e}}} \quad (1 \le a \le 3), \qquad -\frac{H_{ab}}{\sqrt{H_{aa}H_{bb}}} \quad (1 \le a < b \le 3). \tag{58}$$

Substitute (56) and differentiate at $y_i = 1$. Scale invariance makes the all-ones direction null. Permutation invariance makes the Hessian a scalar on the irreducible subspace $\sum_i v_i = 0$. It is therefore enough to compute along $v = (1, -1, 0, 0)$.

After dividing all squared edge lengths by the irrelevant common factor 2^q, one has

$$D_{12} = D_{34} = 1, \qquad D_{13} = D_{14} = \left(1 + \frac{\varepsilon}{2}\right)^q, \qquad D_{23} = D_{24} = \left(1 - \frac{\varepsilon}{2}\right)^q.$$

Expanding the inverse-Gram formulas (58), the six dihedral cosines, in a suitable order, are

$$\begin{aligned} c_1 &= \frac{1}{3} + \left(\frac{5q^2}{27} - \frac{q}{9}\right)\varepsilon^2 + O(\varepsilon^3), \\ c_2 = c_3 &= \frac{1}{3} - \frac{2q}{9}\varepsilon + \frac{q-q^2}{18}\varepsilon^2 + O(\varepsilon^3), \\ c_4 = c_5 &= \frac{1}{3} + \frac{2q}{9}\varepsilon + \frac{q-q^2}{18}\varepsilon^2 + O(\varepsilon^3), \\ c_6 &= \frac{1}{3} - \frac{q^2+q}{9}\varepsilon^2 + O(\varepsilon^3). \end{aligned}$$

Since

$$\arccos\left(\frac{1}{3} + u\right) = \arccos\frac{1}{3} - \frac{3}{2\sqrt{2}}u - \frac{9}{32\sqrt{2}}u^2 + O(u^3),$$

summing the six expansions cancels the linear terms and gives

$$\Sigma_q(1+\varepsilon, 1-\varepsilon, 1, 1) = 6\arccos\frac{1}{3} + \frac{q^2}{6\sqrt{2}}\varepsilon^2 + O(\varepsilon^3).$$

Thus the second derivative in this direction is $q^2/(3\sqrt{2})$. Since $\sum_i v_i^2 = 2$, permutation symmetry on $\sum_i v_i = 0$ yields precisely (57). □

Corollary 6.12 (A uniform neighborhood of the regular metric)**.** *There exists $\delta > 0$ such that, for $1 \le q \le 2$, $\sum_i v_i = 0$, and $\|v\| < \delta$,*

$$\Sigma_q(\mathbf{1} + v) - \Sigma_{\mathrm{reg}} \ge \frac{\|v\|^2}{24\sqrt{2}}.$$

Proof. The Gram matrix is positive definite near the regular metric, uniformly for $q \in [1, 2]$. The dihedral sum is analytic there. Compactness of the exponent interval makes the cubic Taylor remainder in (57) uniform, while its quadratic coefficient is at least $1/(12\sqrt{2})$. Shrinking the neighborhood proves the assertion. □

The two exterior constructions change character inside the open interval. For the positive two-pair path (39), if $1 < q < 2$, then for every $r > 0$,

$$(2r)^q + 2^q = 2^q(1 + r^q) < 4(1 + r)^q. \tag{59}$$

Thus this two-pair symmetric path never reaches a convex planar boundary. For the signed path (42), now $p = 1/k > 1/2$. At the face boundary $r = r_c$, the numerator of z is

$$N(t, r_c) = (1+t)^{2p} - (1 - r_c t)^{2p} - 2^{2p-2}t^{2p}.$$

Here the first difference is $2p(1 + r_c)t + O(t^2)$, whereas $t^{2p} = o(t)$; hence $N(t, r_c) > 0$ for small t. This reverses the sign that created the convex crossing in Lemma 6.4, so that particular boundary-layer proof no longer applies. These observations rule out two proposed counterexample mechanisms, but do not classify all boundary strata and are not a global proof.

A sufficient route to the remaining global bound is to prove Conjecture 6.9, or establish the weaker comparison $\Sigma_y(q) \ge \Sigma_y(1)$. Another route is a boundary and critical-point classification after a scale normalization. Opposite-edge ordering, Regge transformations, and elementary averaging do not by themselves control the sum of six arccosines. Any future proof using those ideas still needs a displayed sign decomposition, a sequence of stated inequalities, or a formally checkable symbolic certificate.

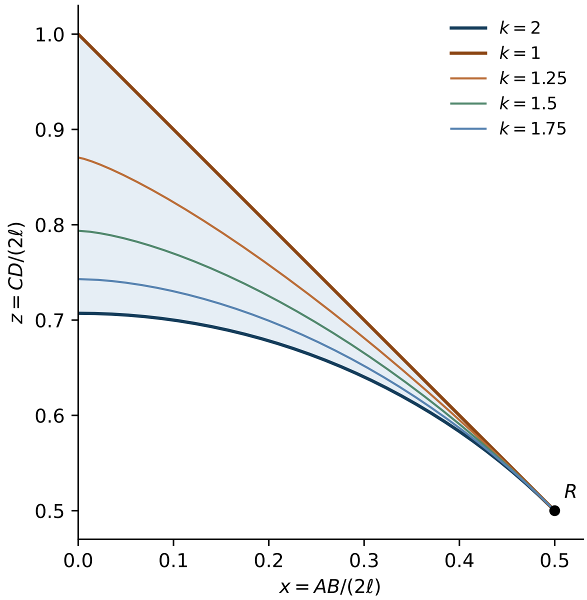

Figure 1: The region $x + z \le 1$, $x^2 + z^2 \ge 1/2$, $0 < x \le z$ used in the two-pair proof. The circular and straight boundaries correspond to $k = 2$ and $k = 1$, respectively. Interior curves show three intermediate exponents; each approaches the regular point $R = (1/2, 1/2)$ as $u/v \to 1$. The figure illustrates the proved inequalities and is not used as evidence for the general four-parameter conjecture.

7 The two-pair family in the closed window

The next result supplies an unconditional part of the Power-k window and proves the stronger exponent-monotonicity conjecture on the same subfamily. Its proof uses only an explicit two-variable angle formula.

Theorem 7.1 (Two-pair window, monotonicity, and exact range)**.** *Let $1 \le k \le 2$ and $y = (u, u, v, v)$, where $u, v > 0$. The six Power-k lengths always form a nondegenerate Euclidean tetrahedron, and*

$$\Sigma(T) \ge \Sigma_{\mathrm{reg}},$$

with equality if and only if $u = v$. For fixed unequal u, v, the dihedral sum for squared edges $(y_i + y_j)^q$ is strictly increasing in q on $[1, 2]$. For each fixed k, the exact range within this two-pair subfamily is

$$\boxed{\left[\Sigma_{\mathrm{reg}},\ 2\pi + 2\arcsin\left(2^{1/k-1}\right)\right).} \tag{60}$$

The upper endpoint is 3π at $k = 1$ and $5\pi/2$ at $k = 2$. For $1 < k < 2$, it is strictly between these two values. This is a subfamily range, not the range of the full Power-k family, whose kite subfamily already approaches 3π.

7.1 Four equal cross-edges

Suppose four cross-edges AC, AD, BC, BD have common length ℓ. Divide all lengths by ℓ and write $AB = 2x$, $CD = 2z$, relabelling the opposite pairs so that $x \le z$. Consider the region

$$0 < x \le z, \qquad x + z \le 1, \qquad x^2 + z^2 \ge \tfrac{1}{2}. \tag{61}$$

These conditions imply $x \le 1/2$ and

$$\sqrt{\tfrac{1}{2} - x^2} \le z \le 1 - x.$$

Since $x, z > 0$ and $x + z \le 1$, one has $x^2 + z^2 < 1$. Thus

$$A = (-x, 0, 0), \quad B = (x, 0, 0), \quad C = (0, w, z), \quad D = (0, w, -z),$$
$$w = \sqrt{1 - x^2 - z^2} > 0$$

defines a nondegenerate tetrahedron with the required six lengths.

The two exceptional dihedrals and the four equal cross-edge dihedrals give

$$\begin{aligned} F(x, z) = 2 \arcsin \frac{z}{\sqrt{1 - x^2}} + 2 \arcsin \frac{x}{\sqrt{1 - z^2}} \\ + 4 \arccos \frac{xz}{\sqrt{(1 - x^2)(1 - z^2)}}. \end{aligned} \tag{62}$$

For example, projection perpendicular to AB gives the angle between the two vectors (w, z) and $(w, -z)$; its cosine is $(w^2 - z^2)/(w^2 + z^2)$. This yields the first term. The second is analogous, and the cross-edge term follows from the adjacent face normals. All inverse trigonometric functions in (62) use principal branches, which give internal angles in $(0, \pi)$.

Lemma 7.2 (Signs of the partial derivatives)**.** *Throughout* (61)*, except at* $(1/2, 1/2)$*,*

$$F_x < 0, \qquad F_z > 0.$$

At $(1/2, 1/2)$ *both derivatives vanish.*

Proof. Differentiating (62), with $w = \sqrt{1 - x^2 - z^2}$, gives

$$F_x = \frac{2(1 - x^2 + xz - 2z)}{(1 - x^2)w}, \qquad F_z = \frac{2(1 - z^2 + xz - 2x)}{(1 - z^2)w}. \tag{63}$$

Let $N_x = 1 - x^2 - (2 - x)z$ and $N_z = 1 - z^2 + xz - 2x$. If $x = 1/2$, the region forces $z = 1/2$. Otherwise $0 < x < 1/2$.

Since $\partial N_z / \partial z = x - 2z < 0$ and $z \le 1 - x$,

$$N_z(x, z) \ge N_z(x, 1 - x) = x(1 - 2x) > 0.$$

To determine the other sign, the polynomial identity

$$\left(\tfrac{1}{2} - x^2\right)(2 - x)^2 - (1 - x^2)^2 = \frac{(1 - 2x)(2x^3 - 3x^2 + 2)}{2} > 0$$

holds for $0 < x < 1/2$: indeed $2x^3 - 3x^2 + 2 > 2 - 3/4 > 0$. All quantities being compared are positive, so

$$z \ge \sqrt{\tfrac{1}{2} - x^2} > \frac{1 - x^2}{2 - x}.$$

Consequently $N_x < 0$. The denominators in (63) are positive, proving the assertions. $\square$

Proposition 7.3 (Four equal cross-edges)**.** *Every tetrahedron in* (61) *satisfies* $F(x, z) \ge \Sigma_{\text{reg}}$*, with equality exactly at* $x = z = 1/2$*. Equivalently, if* $AB = a$*,* $CD = c$ *and the other four edges equal* ℓ*, the sufficient conditions are*

$$a + c \le 2\ell, \qquad a^2 + c^2 \ge 2\ell^2.$$

Proof. Fix $0 < x < 1/2$ and set $z_0(x) = \sqrt{1/2 - x^2}$. By Lemma 7.2,

$$F(x, z) \geq F(x, z_0(x)).$$

The circular boundary lies in the region because $x + z_0(x) \leq 1$, by Cauchy–Schwarz. Along this boundary, $z_0'(x) = -x/z_0(x) < 0$, so

$$\frac{d}{dx} F(x, z_0(x)) = F_x - \frac{x}{z_0(x)} F_z < 0.$$

It follows by continuity at the regular point that

$$F(x, z_0(x)) > F(1/2, 1/2) = 6 \arccos(1/3).$$

The regular point itself gives equality. No assertion about a limiting degenerate tetrahedron is required for this lower-bound proof. $\square$

7.2 Proof of the two-pair theorem

Proof of Theorem 7.1. Exchange the pairs if necessary and put $r = u/v \in (0, 1]$, $e = 1/k \in [1/2, 1]$, and $q = 2e$. After dividing by the cross-edge length $(u + v)^e$,

$$x = \frac{1}{2} \left(\frac{2r}{1 + r} \right)^e, \qquad z = \frac{1}{2} \left(\frac{2}{1 + r} \right)^e. \tag{64}$$

Write $A = 2r/(1 + r)$ and $B = 2/(1 + r)$, so that $A + B = 2$. Concavity of s^e and convexity of s^{2e} give respectively

$$x + z = \frac{A^e + B^e}{2} \leq 1, \qquad x^2 + z^2 = \frac{A^{2e} + B^{2e}}{4} \geq \frac{1}{2}.$$

Thus (61) holds. The coordinates above prove feasibility, and Proposition 7.3 proves the lower bound and equality case.

For exponent monotonicity keep $r < 1$ fixed. Then $0 < A < 1 < B$ and

$$\frac{\partial x}{\partial q} = \frac{x \log A}{2} < 0, \qquad \frac{\partial z}{\partial q} = \frac{z \log B}{2} > 0.$$

Lemma 7.2 therefore gives

$$\frac{d}{dq} \Sigma_y(q) = F_x \frac{\partial x}{\partial q} + F_z \frac{\partial z}{\partial q} > 0.$$

For $r = 1$ the sum is constant at Σ_{reg}.

For the exact range fix e. Formula (64) shows $x'(r) > 0$ and $z'(r) < 0$; hence $dF/dr < 0$ for $0 < r < 1$. At $r = 1$ the tetrahedron is regular. If $e < 1$, then as $r \downarrow 0$,

$$x \longrightarrow 0, \qquad z \longrightarrow 2^{e-1} < 1,$$

and (62) gives

$$F \longrightarrow 2\pi + 2 \arcsin(2^{e-1}).$$

If $e = 1$, then $x + z = 1$ identically. As $x \downarrow 0$,

$$\frac{x}{\sqrt{1 - z^2}} = \sqrt{\frac{x}{2 - x}} \longrightarrow 0, \qquad \frac{xz}{\sqrt{(1 - x^2)(1 - z^2)}} \longrightarrow 0, \qquad \frac{z}{\sqrt{1 - x^2}} \longrightarrow 1.$$

Thus $F \to 3\pi$, agreeing with (60). Continuity and strict monotonicity in r show that every intermediate value is attained and the upper endpoint is not. $\square$

8 The isodynamic family: the full universal range

A tetrahedron is *isodynamic* precisely when its six edge lengths admit positive parameters p_1, p_2, p_3, p_4 such that

$$d_{ij} = p_i p_j \qquad (i \neq j). \tag{65}$$

Equivalently, the products of the lengths in the three pairs of opposite edges are equal. This is the multiplicative metric characterization in [4, Theorem 4]. The following result settles the range completely.

Theorem 8.1 (Exact isodynamic range)**.** *The set of dihedral-angle sums of nondegenerate isodynamic tetrahedra is*

$$\boxed{\{\Sigma(T) : T \ \textit{isodynamic}\} = (2\pi, 3\pi).} \tag{66}$$

Neither endpoint is attained.

Proof. The universal inequalities (1) show at once that the range is contained in $(2\pi, 3\pi)$. We prove the reverse inclusion by joining the regular tetrahedron to each endpoint through explicit isodynamic subfamilies.

For the upper endpoint, take

$$(p_1, p_2, p_3, p_4) = (s, 1, 1, 1).$$

The three edges among vertices $2, 3, 4$ have length 1, and the remaining three have length s. Thus this is exactly the kite of Section 5, with $s > 1/\sqrt{3}$. On the branch $1/\sqrt{3} < s \leq 1$, formula (31) realizes every value in

$$\left[6 \arccos \frac{1}{3}, 3\pi\right).$$

For the lower endpoint, take

$$(p_1, p_2, p_3, p_4) = (t, t, 1, 1) \tag{67}$$

and label the vertices A, B, C, D in this order. Its edge lengths are

$$AB = t^2, \qquad CD = 1, \qquad AC = AD = BC = BD = t.$$

Put

$$h = \frac{1}{2}\sqrt{4t^2 - t^4 - 1}.$$

The coordinates

$$\begin{aligned} A &= (-t^2/2, 0, 0), \quad & B &= (t^2/2, 0, 0), \\ C &= (0, h, 1/2), \quad & D &= (0, h, -1/2) \end{aligned} \tag{68}$$

realize the required lengths whenever

$$\sqrt{2 - \sqrt{3}} < t < \sqrt{2 + \sqrt{3}}. \tag{69}$$

There are three types of internal dihedral angles: one at AB, one at CD, and four equal angles at the cross-edges. Direct scalar products of the inward face normals in (68) give

$$\cos\theta_{AB} = \frac{4t^2 - t^4 - 2}{4t^2 - t^4}, \tag{70}$$

$$\cos\theta_{CD} = \frac{4t^2 - 2t^4 - 1}{4t^2 - 1}, \tag{71}$$

$$\cos\theta_{\times} = \frac{t}{\sqrt{(4 - t^2)(4t^2 - 1)}}. \tag{72}$$

Hence

$$L(t) = \theta_{AB}(t) + \theta_{CD}(t) + 4\theta_\times(t) \tag{73}$$

is continuous throughout (69). At $t = 1$ the tetrahedron is regular, so $L(1) = 6\arccos(1/3)$. If $t_0 = \sqrt{2+\sqrt{3}}$, then (70)–(72) give

$$\theta_{AB} \longrightarrow \pi, \qquad \theta_{CD} \longrightarrow \pi, \qquad \theta_\times \longrightarrow 0 \quad \text{as } t \uparrow t_0.$$

Therefore $L(t) \to 2\pi$. The intermediate value theorem shows that this path realizes every value in $(2\pi, 6\arccos(1/3)]$. Monotonicity is not needed: for any number strictly between the limit and the value at $t = 1$, continuity on a sufficiently long closed subinterval already supplies a preimage. Together with the kite branch, this proves that every number in $(2\pi, 3\pi)$ occurs. The strict universal inequalities show that the endpoints do not occur. □

Remark 8.2. Thus the isodynamic theorem really does reduce to "examples at the two boundaries," but only after three points are checked carefully: both paths must remain inside the isodynamic family, their tetrahedra must remain nondegenerate before the limiting parameter, and the dihedral sum must vary continuously. Equations (65), (68), and (73) supply exactly these checks.

9 Discussion: why the kite coincidence is useful

The equality of the orthocentric range and the kite range has two distinct consequences.

First, because kites are a subfamily, their one-parameter calculation proves sharpness, the unattained upper endpoint, and the realization of all intermediate values. It does *not* prove the lower bound for the larger family; that logical direction is supplied by Theorem 4.4.

Second, the coincidence suggests a stronger comparison statement: perhaps each orthocentric tetrahedron can be symmetrized to a kite without increasing its dihedral sum. The signed metric parametrization $d_{ij}^2 = a_i + a_j$ makes this plausible. If all $a_i \neq 0$ and

$$S = \sum_{i=1}^{4} \frac{1}{a_i},$$

then an inverse-Gram calculation gives, for the dihedral angle between the faces opposite vertices i and j,

$$\cos\theta_{ij} = \frac{\operatorname{sgn}(Sa_ia_j)}{\sqrt{(Sa_i - 1)(Sa_j - 1)}}. \tag{74}$$

When all a_i are positive, the variables

$$y_i = \frac{1}{Sa_i}, \qquad y_i > 0, \qquad \sum_{i=1}^{4} y_i = 1,$$

reduce this to the symmetric formula

$$\cos\theta_{ij} = \sqrt{\frac{y_iy_j}{(1-y_i)(1-y_j)}}. \tag{75}$$

Right-corner cases, for which one additive parameter vanishes, are obtained from these formulas by a limiting argument. A direct symmetrization theorem would be stronger than the range result, but it is unnecessary here: the constant polar sum converts the lower-bound problem into Jensen's inequality for the one-variable function f.

The proof also separates the potentially troublesome obtuse component rather efficiently. Instead of carrying the branch signs through a lengthy six-dihedral calculation, Lemma 4.2 gives $\Omega \geq \beta$ at the exceptional corner, while convexity gives the simple supporting-line estimate $\Omega \geq \beta/2$ at each acute corner. This yields the stronger non-acute bound $19\pi/8$ and leaves equality possible only in the all-acute component.

The Power-k family exhibits two rigorously different exterior regimes. The regular lower bound is proved here at $k = 2$, while explicit convex-planar degenerations lower the infimum to 2π for $k < 1$ and for $k > 2$. In the first regime the degeneration already occurs in the positive branch; in the second our construction uses one negative Power-k parameter. The full-family bound for $1 < k < 2$ remains open, although Theorem 7.1 settles the complete two-pair subfamily, as does the unexpanded global step at $k = 1$ noted in Remark 6.6. Proposition 6.8 and Conjecture 6.9 isolate a unified route to both questions.

The families studied here illustrate several genuinely different proof mechanisms. For orthocentric tetrahedra, a constant sum for a secondary angle measure creates the decisive convexity argument. For isodynamic tetrahedra, the universal theorem plus explicit endpoint paths settles the range, even though the sum need not be monotone along the lower path. For isogonic tetrahedra, exact formulas and the kite subfamily settle sharpness and the upper part of the range, but they do not yet exclude a hidden value below the regular one. Conjecture A.2 isolates that missing step in a form suitable for either a human factorization or a formally checkable symbolic certificate.

10 Reproducible computations

The proofs of the main range theorems are analytic. The accompanying supplement provides independent high-precision checks of the two-pair angle formula, its derivatives, and the regular-point expansion, together with a rigorous interval computation for the following obstruction to an unrestricted averaging argument.

At $q = 3/2$, let $y = (1, 1, 4, 8)$ and $\widetilde{y} = (1, 1, 41/10, 79/10)$. Both squared-distance matrices $D_{ij} = (y_i + y_j)^{3/2}$ define nondegenerate tetrahedra. Arb interval arithmetic [6], starting with exact rational inputs, certifies

$$0.001467 < \Sigma_{\widetilde{y}}(3/2) - \Sigma_y(3/2) < 0.001468. \tag{76}$$

Thus bringing two parameters closer while preserving their sum can increase Σ. This rules out unrestricted pairwise averaging as a descent argument, but does not refute exponent monotonicity or a specially ordered averaging scheme. The script `certify_averaging.py` verifies positive leading principal minors of both Gram matrices and encloses all six principal dihedral angles. This finite certified example is distinct from the exploratory sampling provided in the supplement.

Appendix B gives the derivatives used in the exploratory code. Sampling is not exhaustive and establishes no global inequality.

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Supplementary material

The accompanying archive contains the complete LaTeX source, figure source, high-precision verification scripts, an Arb certificate for (76), analytic derivative routines, a seeded exploratory search, recorded outputs, dependency versions, and build instructions. No supplementary computation is needed to complete the analytic proofs of the stated range theorems.

A A comparison family: isogonic tetrahedra

The isogonic family is included here for comparison, not as another member of the Power-k chain. The exact formulas below are proved, but the regular lower bound remains conditional on Conjecture A.2. This appendix should therefore be read as a self-contained open problem rather than as part of the completed Power-k classification.

A tetrahedron is *isogonic* precisely when its squared edge lengths can be written

$$d_{ij}^2 = p_i^2 + p_i p_j + p_j^2, \qquad p_1, p_2, p_3, p_4 > 0; \tag{77}$$

see [4, Theorem 4]. In contrast with the orthocentric case, the following calculation exposes the whole dihedral sum without a choice of branches or auxiliary solid-angle measures.

For complementary indices $\{i, j, k, l\} = \{1, 2, 3, 4\}$, set

$$q_{ij} = p_i^2 + p_i p_j + p_j^2, \quad E = \sum_{1 \le r < s \le 4} p_r p_s, \quad h_i = p_j p_k + p_j p_l + p_k p_l. \tag{78}$$

Proposition A.1 (Isogonic half-angle formula). *Let θ_{ij} be the internal dihedral angle between the faces opposite vertices i and j; equivalently, it is the angle at edge kl. Then*

$$\sin^2 \frac{\theta_{ij}}{2} = \frac{p_i p_j q_{kl}}{h_i h_j}, \tag{79}$$

$$\tan^2 \frac{\theta_{ij}}{2} = \frac{p_i p_j q_{kl}}{p_k p_l E}. \tag{80}$$

Consequently

$$\boxed{\Sigma_{\mathrm{igo}}(p_1, p_2, p_3, p_4) = 2 \sum_{1 \le i < j \le 4} \arctan \sqrt{\frac{p_i p_j q_{kl}}{p_k p_l E}}.} \tag{81}$$

Proof. Heron's formula applied to the face opposite vertex i simplifies to

$$A_i = \frac{\sqrt{3}}{4} h_i. \tag{82}$$

The determinant of the edge Gram matrix based at any vertex simplifies to

$$36V^2 = \frac{9}{4} p_1 p_2 p_3 p_4 E, \qquad V^2 = \frac{1}{16} p_1 p_2 p_3 p_4 E. \tag{83}$$

A direct cofactor calculation for the angle between two face normals then gives

$$\cos \theta_{ij} = 1 - \frac{2 p_i p_j q_{kl}}{h_i h_j}. \tag{84}$$

Using $\sin^2(\theta/2) = (1 - \cos\theta)/2$ proves (79). Finally, direct expansion gives the useful identity

$$h_i h_j - p_i p_j q_{kl} = p_k p_l E. \tag{85}$$

Dividing (79) by its complementary cosine-square proves (80), and summing the six principal half-angles proves (81). □

The formula immediately explains why kites are central here. If $(p_1, p_2, p_3, p_4) = (a, b, b, b)$, then the base edges have squared length $3b^2$ and the three lateral edges have squared length $a^2 + ab + b^2$. Writing $s = a/b$, the squared lateral-to-base ratio is

$$t^2 = \frac{s^2 + s + 1}{3}. \tag{86}$$

As s runs over $(0, \infty)$, t runs over $(1/\sqrt{3}, \infty)$. Therefore the isogonic family contains every equilateral-base kite, and in particular it realizes

$$\left[6 \arccos \frac{1}{3}, 3\pi\right). \tag{87}$$

The unresolved direction is to prove that no isogonic tetrahedron lies below the regular value.

Formula (81) suggests the following precise reduction.

Conjecture A.2 (Ordered kite-reduction inequality)**.** *Suppose* $0 < p_1 \le p_2 \le p_3 \le p_4$ *and put*

$$m = \frac{p_1 + p_2 + p_3}{3}.$$

Then

$$\Sigma_{\mathrm{igo}}(p_1, p_2, p_3, p_4) \ge \Sigma_{\mathrm{igo}}(m, m, m, p_4). \tag{88}$$

Equality should hold only when $p_1 = p_2 = p_3$*.*

Theorem A.3 (Conditional exact isogonic range)**.** *If Conjecture A.2 holds, then*

$$\{\Sigma(T) : T \ \textit{isogonic}\} = \left[6 \arccos \frac{1}{3}, 3\pi\right), \tag{89}$$

and equality at the lower endpoint occurs only for the regular tetrahedron.

Proof. After relabelling, the parameters may be ordered. The right-hand side of (88) is the sum of an equilateral-base kite and is therefore at least its regular value by (32). Equality in the conjecture first forces $p_1 = p_2 = p_3$; equality in the kite bound then forces $p_4 = m$, so all four parameters are equal and the tetrahedron is regular. The upper bound is universal, and (87) already supplies every value from the regular value to 3π. $\square$

The ordering in Conjecture A.2 is essential to its formulation. Neither the half-angle formula nor ordinary pairwise averaging provides the required sign. We therefore record the reduction only as an open problem: a proof must exhibit a readable chain of inequalities or a formally checkable symbolic certificate.

B Inverse-Gram derivatives

Let $D_{ii} = 0$, $D_{ij} = (y_i + y_j)^q$ for $i \ne j$, and define

$$G_{ab} = \frac{D_{1,a+1} + D_{1,b+1} - D_{a+1,b+1}}{2}, \quad H = G^{-1}, \quad J = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K = JHJ^T.$$

The internal dihedral between faces opposite i, j has cosine

$$c_{ij} = -\frac{K_{ij}}{\sqrt{K_{ii}K_{jj}}}.$$

For differentiation in q, use $D'_{ij} = D_{ij}\log(y_i + y_j)$ off the diagonal, $D'_{ii} = 0$, and

$$H' = -HG'H, \qquad K' = -JHG'HJ^T,$$

$$c'_{ij} = -\frac{K'_{ij}}{\sqrt{K_{ii}K_{jj}}} - \frac{c_{ij}}{2}\left(\frac{K'_{ii}}{K_{ii}} + \frac{K'_{jj}}{K_{jj}}\right), \qquad \Sigma'_y(q) = -\sum_{i<j}\frac{c'_{ij}}{\sqrt{1-c_{ij}^2}}.$$

These identities remove finite-difference noise from numerical exploration. No general sign certificate for the last expression has been established here.

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