

Reciprocal Periods and Primality

A period–primality toolbox and the rarity of small-index pseudoprimes

Extended arXiv version

Abstract

Suppose that the exact reciprocal period

$$L = L_b(n) = \text{ord}_n(b)$$

is known for an odd integer n and a base b coprime to n . How much information about the primality of n is already contained in L ? This expository question is the starting point of the present extended version. We assemble a period–primality toolbox connecting multiplicative order with Lucas’ criterion, Fermat and strong pseudoprimality, Midy’s property, overpseudoprimes, prime-only tests for total Midy, q -Midy congruence forcing, cyclotomic factors, and $N \pm 1$ certificates of Pocklington and Brillhart–Lehmer–Selfridge type. The toolbox assumes that the exact order is already available; it is not proposed as a general-purpose primality algorithm.

The particularly important case $L \mid n - 1$ leads to the reciprocal-period index

$$d = \frac{n - 1}{L}.$$

For composite n this is Somer’s Fermat d -pseudoprime condition; for prime moduli the same d is the standard residual index. The special $\kappa = 1$ composite stratum is the prescribed Carmichael-index problem. Somer proved fixed- d finiteness and classified the nontrivial-base cases for $1 \leq d \leq 8$. Our spectral investigation gives two direct primality consequences. First, for $d \leq 64$ an odd integer with $L = (n - 1)/d$ is prime except for the explicitly certified composite table, which contains 234 distinct integers. Second, if

$$S(d) = 1 + \lfloor \log_2(d - 1) \rfloor, \quad M(d) = 2^{S(d)-1},$$

then every composite spectral point satisfies

$$n < (dS(d))^{2M(d)}.$$

Consequently, for every fixed $\varepsilon > 0$, all sufficiently large n with

$$d \leq \left(\frac{1}{2} - \varepsilon \right) \frac{\log n}{\log \log n}$$

are prime.

The base-independent genuine spectrum is also shown to be a partition of all odd composite integers: each odd composite appears exactly once at its genuine index

$$d_\lambda(n) = \frac{n - 1}{\gcd(n - 1, \lambda(n))},$$

while all later appearances are inherited. We certify the complete inclusive spectrum through $d = 64$, study the fixed-base prime spectrum via the Wagstaff–Moree near-primitive-root densities, and prove under GRH that the same range contains at least 98.2709160681865... % of the prime mass for every positive non-perfect-power integer base. We also introduce the counting functions $e(d)$ and $g(d)$. The κ - H search decomposition has $\frac{1}{2}d(\log d)^2 + O(d \log d)$ overlap-resolved branches, while collapsing to the Diophantine master equation gives a relaxed

parameter count of only $d \log d + O(d)$ candidate pairs. A divisor-pair identity for the last two prime factors then yields the uniform counting estimate

$$\log e(d) \leq \frac{d}{2}(\log d + \log \log d + O(1)).$$

In particular the one-, two-, and three-distinct-prime strata satisfy $e_1(d) \leq d^{o(1)}$ and $e_2(d), e_3(d) \leq d^{1+o(1)}$. Under a natural uniform sampling model on a spectral line, for fixed d we prove

$$\Pr(n \text{ prime} \mid d) = 1 - e(d)\phi(d)\frac{\log X}{X} + o\left(\frac{\log X}{X}\right).$$

More strongly, for every fixed $0 < \varepsilon < 2$ this probability tends to 1 uniformly throughout

$$1 \leq d \leq (2 - \varepsilon)\frac{\log X}{\log \log X}.$$

Thus the deterministic certainty range with leading constant $1/2$ is complemented by a substantially wider high-probability range, in the stated sampling model, with leading constant 2. Four proposed OEIS sequences are printed in full through $d = 64$.

Keywords. Fermat d -pseudoprimes; residual index; near-primitive roots; Carmichael index; reciprocal periods; Carmichael function; Miller–Rabin; computer-assisted proof.

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1 Introduction

The elementary decimal identity

$$\frac{1}{7} = 0.\overline{142857}$$

contains a surprisingly strong arithmetic statement. The six-digit period means

$$\text{ord}_7(10) = 6 = 7 - 1,$$

and the classical Lucas criterion immediately proves that 7 is prime. More generally, if the exact period

$$L = L_b(n) = \text{ord}_n(b)$$

is known, then the natural question is:

How much information about primality is encoded in the exact period L ?

This question, rather than a pre-existing pseudoprime classification, was the motivation that led to the present project. The first goal was to collect in one place the most useful consequences of knowing L : direct Lucas-type criteria, Fermat and strong-pseudoprime implications, Midy decompositions, overpseudoprimes, cyclotomic information, and positive $N - 1$ certificates of

Pocklington and Brillhart–Lehmer–Selfridge type. The resulting toolbox is given in Sections 5 and 6 with explicit examples.

A second question emerged naturally. When

$$L \mid n - 1,$$

write

$$n - 1 = dL, \quad d = \frac{n - 1}{L}. \quad (1)$$

For a prime p , d is the index of the cyclic subgroup generated by b in $(\mathbb{Z}/p\mathbb{Z})^\times$. A small value of d therefore says that the known period is close to the maximal prime value $p - 1$. It is natural to ask whether sufficiently small d forces primality.

For example,

$$91 = 7 \cdot 13, \quad \text{ord}_{91}(10) = 6,$$

so $91 - 1 = 15 \cdot 6$. Thus a composite can have $L \mid n - 1$, but here the index is already $d = 15$. The present paper studies systematically how small such an index can be for a composite.

The rediscovery of Somer’s d -pseudoprimes

During the development of this project, after the reciprocal index (1) and the first small-index arguments had already been obtained independently, an AI-assisted literature search located Lawrence Somer’s 1985 dissertation and his 1989 paper *On Fermat d -pseudoprimes* [32, 33]. Somer had introduced precisely the composite condition

$$\text{ord}_N(a) = \frac{N - 1}{d},$$

proved that for each fixed d only finitely many such composites exist, and classified the nontrivial-base cases for $1 \leq d \leq 8$. Our parameter d is therefore Somer’s parameter.

Somer’s work was not wholly forgotten, but the independent uptake that we have been able to locate is surprisingly concentrated. Two examples are particularly clear. Ribenboim did more than cite the paper: he devoted Section VIII.E, entitled “Somer Pseudoprimes,” of *The New Book of Prime Number Records* to the subject [29, p. 117]. A decade later, Pinch’s extended abstract *Carmichael numbers with small index* opens by stating that Somer had proved a result implying that the Carmichael index $(N - 1)/\lambda(N)$ tends to infinity, then gives a new proof and an enumeration algorithm for bounded Carmichael index [25]. Pinch’s 2006 tabulation through 10^{18} also cites Somer’s Fermat d -pseudoprime paper [23].

A separate line of continuation came from Somer and his collaborators. Somer developed the analogous Lucas d -pseudoprime theory [34]; Carlip, Jacobson and Somer connected the same methods with Lehmer’s problem and multiperfect numbers [6]; and later work developed Lucas d -pseudoprimes and Carmichael–Lucas numbers further [7].

The Carlip–Jacobson–Somer paper is particularly relevant to the Fermat side of the present work. It develops Somer’s methods in a unified setting and, for a Fermat d -pseudoprime with t distinct prime factors, proves the factor-count restriction

$$t < \log_2 d + 1,$$

together with an explicit bound on repeated prime powers. These are direct antecedents of part of the structural theory below. Our refinement retains the additional overlap parameter $\kappa = \lambda(n)/L$: the exact divisibilities $K \mid \kappa$ and $2^{s-1}\kappa \mid j$ sharpen the factor-count and repeated-power restrictions, while the uniform closed size bound, the last-two-prime divisor-pair identity, and the counting estimate for $e(d)$ are additional quantitative consequences developed here.

Our targeted literature search has not revealed a comparably sustained independent literature using the *variable-base Fermat d -spectrum* as a primary organizing framework. We therefore

describe Somer's Fermat theory as under-recognized rather than forgotten. This is not a bibliometric census or a numerical citation claim: database coverage of an older conference chapter is imperfect, and absence from a search is not evidence of absence. What is relevant for the present paper is that the same reciprocal index arose independently here from a different motivation, namely the information content of a known period.

This convergence of two routes is mathematically useful. Somer's viewpoint starts from generalized Fermat pseudoprimes. Our route starts from reciprocal periods and primality. Both single out

$$d = \frac{n-1}{L}$$

as the natural coordinate measuring the deficiency of the period from the Lucas extremal value $L = n - 1$.

Two primality theorems produced by the spectral investigation

The spectral part of the paper yields two answers to the motivating primality question.

P1. Exact low-index criterion. If n is odd, $\gcd(b, n) = 1$, $L = \text{ord}_n(b)$, and

$$L = \frac{n-1}{d}, \quad d \leq 64,$$

then n is prime unless it occurs in the certified exceptional table of Section 9. Across all 64 rows there are 234 distinct composite exceptions in 351 spectral incidences.

P2. Uniform size criterion. For $d \geq 2$ put

$$S(d) = 1 + \lfloor \log_2(d-1) \rfloor, \quad M(d) = 2^{S(d)-1}.$$

Theorem 7.8 proves that every composite spectral point obeys

$$n < (dS(d))^{2M(d)}.$$

Hence

$$n \geq (dS(d))^{2M(d)}, \quad L = (n-1)/d \implies n \text{ is prime}.$$

Asymptotically, for each fixed $\varepsilon > 0$,

$$d \leq \left(\frac{1}{2} - \varepsilon \right) \frac{\log n}{\log \log n}$$

forces primality for all sufficiently large n .

The first statement is finite and exact; the second is uniform in d . The coefficient $1/2$ in the asymptotic statement is the worst leading coefficient of our uniform bound, not a claim about the true optimal boundary.

They are complemented by a statistical consequence once a sampling model is specified. If one samples uniformly from all integers up to X lying on one fixed existential line d , Theorem 11.6 gives

$$\Pr_X(n \text{ prime} \mid d) = 1 - e(d)\phi(d)\frac{\log X}{X} + o\left(\frac{\log X}{X}\right).$$

Thus fixed small index is not merely compatible with primality: in this precise line-sampling sense it becomes asymptotically overwhelming. This probability statement is not one of the two deterministic primality theorems above and is not an intrinsic Bayesian probability attached to a single integer.

Prime–pseudoprime spectral separation

When the base is allowed to vary, the prime line at a fixed d is simply

$$\mathcal{P}(d) = \{p \text{ prime} : p \equiv 1 \pmod{d}\},$$

and is therefore infinite. Somer’s theorem says that the corresponding composite line is finite. The present work strengthens the comparison in two directions. On the composite side we derive explicit structural bounds and certify the complete spectrum through $d = 64$. On the prime side we compare the same index range with the fixed-base near-primitive-root densities of Wagstaff and Moree. Under GRH, at least 98.2709160681865... % of the prime mass lies in $d \leq 64$ for every positive integer base that is not a perfect power, while the inclusive composite union through 64 contains 234 integers.

A second structural point is equally important. Allowing the trivial orders 1 and 2 gives an inclusive group-theoretic spectrum in which *every odd composite integer appears at a unique genuine index*. Thus the genuine spectrum is not merely a collection of exceptional examples: it is a partition of the odd composites. Later spectral occurrences are inherited from the genuine one.

Guide to the extended version

This arXiv version is deliberately more expository than the compact working paper. Section 2 introduces multiplicative order, group exponent, the Carmichael function, Chinese-remainder decomposition and Sylow components. Sections 5 and 6 then develop the period–primality toolbox with examples. The spectral structure, genuine spectrum and uniform bounds follow next. Somer’s small-index classification is reproved uniformly, and the exact computation through $d = 64$ is tied to a frozen supplement combining the original Julia certification through 33 with the exact extension through 64. The final sections treat counting functions, conditional primality probabilities, Artin’s constant and fixed-base prime spectra, and four natural OEIS sequences.

Notation at a glance

The following symbols recur throughout the paper.

symbol	meaning
n	an odd integer under primality investigation
b	a base with $\gcd(b, n) = 1$
$L = L_b(n) = \text{ord}_n(b)$	the exact multiplicative order of b modulo n , equivalently the reciprocal period
$\phi(n)$	Euler’s totient, the order of $(\mathbb{Z}/n\mathbb{Z})^\times$
$\lambda(n)$	Carmichael’s function, the exponent of $(\mathbb{Z}/n\mathbb{Z})^\times$
$d = (n - 1)/L$	the spectral or reciprocal-period index, when $L \mid n - 1$
$L_{\max}(n)$	$\gcd(n - 1, \lambda(n))$, the largest element order compatible with $n - 1$
$d_\lambda(n)$	$(n - 1)/L_{\max}(n)$, the unique genuine spectral index of a composite
$\mathcal{E}(d)$	odd composites occurring on the inclusive existential line d
$\mathcal{G}(d)$	odd composites whose genuine index equals d
$e(d), g(d)$	$ \mathcal{E}(d) $ and $ \mathcal{G}(d) $

2 Reciprocal periods and finite unit groups

2.1 Period length as multiplicative order

Let $n > 1$, $b \geq 2$ and $\gcd(b, n) = 1$. The residue class of b is then a unit modulo n . Its *multiplicative order*

$$\text{ord}_n(b)$$

is the least positive integer L for which $b^L \equiv 1 \pmod{n}$. Equivalently, L is the period length of the purely recurring base- b expansion of $1/n$. We write

$$L_b(n) = \text{ord}_n(b).$$

The units modulo n form the finite abelian group

$$U(n) = (\mathbb{Z}/n\mathbb{Z})^\times.$$

Its order is Euler's totient

$$|U(n)| = \phi(n).$$

Thus Lagrange's theorem already gives

$$L_b(n) \mid \phi(n).$$

The sharper universal divisor is the Carmichael function $\lambda(n)$.

2.2 Exponent of a finite group and the Carmichael function

Definition 2.1 (Exponent). *If G is a finite group, its exponent is*

$$\exp(G) = \text{lcm}\{\text{ord}(g) : g \in G\}.$$

Equivalently, it is the least positive integer e such that $g^e = 1$ for every $g \in G$.

The Carmichael function is precisely the exponent of the unit group:

$$\boxed{\lambda(n) = \exp U(n).}$$

Consequently

$$L_b(n) \mid \lambda(n) \mid \phi(n). \tag{2}$$

The first divisibility says that every individual reciprocal period divides a single base-independent invariant of n .

For later calculations it is useful to recall the explicit prime-power formula. If p is odd, then

$$\lambda(p^a) = \phi(p^a) = p^{a-1}(p-1).$$

For powers of 2,

$$\lambda(2) = 1, \quad \lambda(4) = 2, \quad \lambda(2^a) = 2^{a-2} \quad (a \geq 3).$$

If $n = \prod_i p_i^{a_i}$, the Chinese remainder theorem gives

$$U(n) \cong \prod_i U(p_i^{a_i}),$$

and therefore

$$\lambda(n) = \text{lcm}_i \lambda(p_i^{a_i}).$$

2.3 Sylow components and attainable element orders

A finite abelian group separates prime by prime. Its *Sylow q -component* G_q consists of the elements whose orders are powers of the prime q , and

$$G \cong \prod_{q||G|} G_q.$$

This simple decomposition is the reason that, in a finite abelian group, there are no gaps among the divisors of the exponent.

Lemma 2.2 (Orders dividing the exponent). *Let G be a finite abelian group of exponent e . Every positive divisor of e occurs as the order of an element of G . Consequently, for every $n > 1$ and every $L \geq 1$,*

$$\exists b : \text{ord}_n(b) = L \iff L \mid \lambda(n).$$

Proof. Write

$$e = \prod_q q^{a_q}.$$

Because the exponent of the q -Sylow component is q^{a_q} , that component contains an element of order q^{a_q} ; taking powers produces elements of every order q^j with $0 \leq j \leq a_q$. If $m = \prod_q q^{j_q} \mid e$, choose in each G_q an element of order q^{j_q} . Their product, under the direct-product decomposition, has order m . Applying this to the abelian group $U(n)$ proves the second statement. $\square$

Remark 2.3. *This lemma is stronger than Lagrange's theorem. In an arbitrary finite group, a divisor of $|G|$ need not occur as an element order. What matters here is that the unit group is abelian and that we use divisors of its exponent, not arbitrary divisors of its cardinality.*

2.4 Spectral lines

Whenever $L_b(n) \mid n - 1$ we define

$$d = \frac{n - 1}{L_b(n)}.$$

For primes, $U(p)$ is cyclic of order $p - 1$, and d is literally the index of the subgroup generated by b :

$$d = [U(p) : \langle b \rangle].$$

For this reason we call d the *reciprocal-period index* or *spectral index*.

Definition 2.4 (Existential prime and composite spectra). *For $d \geq 1$, define*

$$\mathcal{P}(d) = \left\{ p \text{ prime} : \exists b \geq 2, \gcd(b, p) = 1, \text{ord}_p(b) = \frac{p - 1}{d} \right\},$$

and

$$\mathcal{E}(d) = \left\{ n \text{ odd composite} : \exists b \geq 2, \gcd(b, n) = 1, \text{ord}_n(b) = \frac{n - 1}{d} \right\}.$$

For comparison with Somer's convention we also write

$$\mathcal{E}^*(d) = \left\{ n \in \mathcal{E}(d) : \exists b \not\equiv \pm 1 \pmod{n}, \text{ord}_n(b) = \frac{n - 1}{d} \right\}.$$

Thus $\mathcal{E}^*(d)$ is the nontrivial-base Fermat d -pseudoprime line, whereas $\mathcal{E}(d)$ is the inclusive group-theoretic spectral line. The base is allowed to depend on n .

Theorem 2.5 (Prime spectral line). *For every $d \geq 1$,*

$$\mathcal{P}(d) = \{p \text{ prime} : p \equiv 1 \pmod{d}\}.$$

Hence $\mathcal{P}(d)$ is infinite. Its relative density among primes is $1/\phi(d)$.

Proof. The cyclic group $U(p)$ contains an element of order $(p-1)/d$ if and only if $d \mid p-1$. Dirichlet's theorem gives infinitude and the prime number theorem for arithmetic progressions gives the relative density. $\square$

Proposition 2.6 (Composite spectral line). *For an odd composite n ,*

$$n \in \mathcal{E}(d) \iff d \mid n-1 \text{ and } \frac{n-1}{d} \mid \lambda(n).$$

Proof. Apply Lemma 2.2 with $L = (n-1)/d$. $\square$

3 Genuine and inherited spectral indices

For a fixed composite n , one integer may occur on more than one spectral line. The natural distinguished line is determined by the largest order compatible with $n-1$.

Definition 3.1. *Define*

$$L_{\max}(n) = \gcd(n-1, \lambda(n)), \quad d_{\lambda}(n) = \frac{n-1}{L_{\max}(n)}.$$

We call $d_{\lambda}(n)$ the genuine spectral index of n .

Theorem 3.2 (Spectral inheritance). *For an odd composite n ,*

$$\text{Spec}_L(n) := \{D : n \in \mathcal{E}(D)\} = \{d_{\lambda}(n)r : r \mid L_{\max}(n)\}.$$

In particular, every composite has one unique genuine spectral index, namely the least index at which it occurs.

Proof. Every admissible order is a positive divisor of $L_{\max}(n)$. If

$$L = \frac{L_{\max}(n)}{r}, \quad r \mid L_{\max}(n),$$

then

$$\frac{n-1}{L} = d_{\lambda}(n)r.$$

Conversely, Lemma 2.2 shows that every divisor L of $L_{\max}(n)$ occurs as the order of some unit modulo n . $\square$

This theorem explains why the same pseudoprime reappears at higher spectral indices. The later occurrences are inherited from its genuine line.

Corollary 3.3 (The genuine spectrum partitions the odd composites). *Let*

$$\mathcal{G}(d) = \{n \text{ odd composite} : d_{\lambda}(n) = d\}.$$

Then

$$\boxed{\{n \geq 3 : n \text{ odd composite}\} = \bigsqcup_{d \geq 1} \mathcal{G}(d).}$$

Thus every odd composite occurs somewhere in the genuine spectrum and occurs there exactly once.

Proof. For every odd composite n , the integer $L_{\max}(n) = \gcd(n-1, \lambda(n))$ is positive and divides $n-1$, so $d_\lambda(n)$ is a positive integer. Theorem 3.2 shows that it is the unique least spectral index of n . Hence n belongs to exactly one set $\mathcal{G}(d)$. $\square$

This partition property is one reason the genuine spectrum is more than a list of exceptional pseudoprimes. It supplies a canonical base-independent coordinate to every odd composite integer. The inclusive convention is essential: the genuine witnessing order can be 1 or 2.

4 A hierarchy of L -, λ -, and ϕ -spectra

The same denominator viewpoint can be applied directly to arithmetic functions.

Definition 4.1. For $f \in \{\lambda, \phi\}$ define

$$d_f(n) = \frac{n-1}{\gcd(n-1, f(n))}.$$

Equivalently, $d_f(n)$ is the denominator of $f(n)/(n-1)$ when written in lowest terms.

Then

$$d_\lambda(n) = \frac{n-1}{L_{\max}(n)}.$$

Thus the genuine L -spectrum is the λ -spectrum.

Since $\lambda(n) \mid \phi(n)$,

$$d_\phi(n) \mid d_\lambda(n).$$

Writing

$$H(n) = \frac{\phi(n)}{\lambda(n)},$$

one also has

$$\frac{d_\lambda(n)}{d_\phi(n)} \mid H(n).$$

The classical Lehmer problem is the extreme ϕ -spectral question

$$\phi(n) \mid n-1.$$

The original problem and classical restrictions on a hypothetical composite solution are discussed in [18, 26]. The more general ϕ -spectrum therefore contains Lehmer-type questions as a special case. In contrast, the L - and λ -spectra considered here admit finite spectral lines by the bounds proved below.

5 Preliminaries for the period–primality toolbox

This section records several elementary reductions that will be used both in the proofs and in the extended toolbox. They are stated in a form adapted to an exact period $L = \text{ord}_n(b)$.

5.1 The repetend as an integer

Suppose $\gcd(b, n) = 1$ and $L = \text{ord}_n(b)$. If the repetend of $1/n$ in base b is regarded as an L -digit integer R (allowing leading zeroes), then

$$R = \frac{b^L - 1}{n}. \tag{3}$$

Indeed,

$$\frac{1}{n} = 0.\overline{R}_b = \frac{R}{b^L - 1}.$$

This simple identity is the bridge between multiplicative order and Midy's block-sum phenomena.

5.2 A reduction of Somer

For

$$n = \prod_{i=1}^s p_i^{a_i},$$

define

$$\lambda_0(n) := \text{lcm}_{p|n}(p-1) = \text{lcm}(p_1-1, \dots, p_s-1). \quad (4)$$

This is Somer's $\lambda'(n)$.

Lemma 5.1 (Somer's order reduction). *If*

$$L = \text{ord}_n(b) \mid n-1,$$

then

$$\gcd(n, L) = 1, \quad L \mid \lambda_0(n).$$

In particular, if $n-1 = dL$, then also $\gcd(n, d) = 1$.

Proof. The coprimality assertions follow from $L \mid n-1$ and $d \mid n-1$. For each prime power $p_i^{a_i} \parallel n$, the local order $\text{ord}_{p_i^{a_i}}(b)$ divides $p_i^{a_i-1}(p_i-1)$ and also divides L . Since L is coprime to p_i , the local order actually divides p_i-1 . Taking the least common multiple over i gives $L \mid \lambda_0(n)$. This is essentially Lemma 1 of Somer [33]. $\square$

Remark 5.2. *The standard relation $L \mid \lambda(n)$ is valid without the assumption $L \mid n-1$. Lemma 5.1 is stronger precisely in the spectral situation: prime-power factors coming from $p_i^{a_i-1}$ cannot contribute to L because L is coprime to n .*

5.3 Block splitting and a cyclotomic criterion

Let $r > 1$ divide L and put $h = L/r$. Split the L digits of the repetend into r consecutive blocks of h digits, represented by integers $A_0, \dots, A_{r-1}$ with $0 \leq A_i < b^h$. Since $b^h \equiv 1 \pmod{b^h-1}$,

$$R \equiv A_0 + \dots + A_{r-1} \pmod{b^h-1}.$$

Combining this with (3) gives the following useful special form of Midy's property. For broader treatments of Midy-type block criteria and their order-theoretic formulations, see [13, 19, 20, 12].

Proposition 5.3 (Midy block criterion for $1/n$). *Let $r \mid L$, $r > 1$, and $h = L/r$. The sum of the r equal-length blocks is divisible by b^h-1 if and only if*

$$n \mid 1 + b^h + b^{2h} + \dots + b^{(r-1)h}. \quad (5)$$

If $r = q$ is prime, this is

$$n \mid \Phi_q(b^h).$$

For prime $n = p$, condition (5) holds for every $r > 1$ dividing L .

Proof. By (3),

$$R = \frac{(b^h-1)(1 + b^h + \dots + b^{(r-1)h})}{n}.$$

Thus $b^h-1 \mid R$ exactly when n divides the geometric sum. If $n = p$ is prime, then b^h has exact order r modulo p ; hence the geometric sum vanishes modulo p . $\square$

For $r = 2$, the two blocks have sum exactly b^h-1 whenever the Midy condition holds, because their sum is a positive multiple of b^h-1 but is strictly less than $2(b^h-1)$. Moreover,

$$\text{two-block Midy property} \iff b^{L/2} \equiv -1 \pmod{n}. \quad (6)$$

5.4 Two prime-only tests for the total Midy property

The phrase *total Midy property* means that the block condition of Proposition 5.3 holds for every divisor $r > 1$ of the exact period L . At first sight this asks for many block divisors and, in the order-theoretic formulation, for every nontrivial divisor of the modulus. Both requirements admit a prime-only reduction.

Proposition 5.4 (Prime-only characterizations of total Midy). *Let $N > 1$ be odd, $\gcd(N, b) = 1$, and $L = \text{ord}_N(b)$. The following are equivalent.*

- (i) *the Midy block condition holds for every $r > 1$ dividing L ;*
- (ii) *it is enough to check the block condition only for every prime $q \mid L$;*
- (iii)
$$\text{ord}_p(b) = L \quad \text{for every prime } p \mid N;$$
- (iv)
$$\text{ord}_D(b) = L \quad \text{for every divisor } D > 1 \text{ of } N.$$

Thus total Midy can be checked on the prime divisors of the period, or, equivalently, on the prime divisors of the modulus; there is no need to check all divisors in either family.

Proof. The upward-closure property of the Midy set says that if $r_1 \mid r_2 \mid L$ and the property holds for r_1 , then it holds for r_2 . Hence checking all prime $q \mid L$ already checks every divisor $r > 1$ of L . The equivalence of (i) and (iii) is Theorem 2.10 of [9] (with the same result developed from the Midy-set characterizations in the preceding results of that paper).

It remains only to observe that (iii) and (iv) are elementarily equivalent. If $p^a \parallel N$, reduction modulo p gives

$$\text{ord}_p(b) \mid \text{ord}_{p^a}(b),$$

whereas $p^a \mid N$ gives

$$\text{ord}_{p^a}(b) \mid \text{ord}_N(b) = L.$$

Thus $\text{ord}_p(b) = L$ forces $\text{ord}_{p^a}(b) = L$. The order modulo an arbitrary divisor $D > 1$ is the least common multiple of its prime-power local orders, so it is again L . The converse follows by taking $D = p$. $\square$

A caution is useful. Merely checking that the orders modulo the prime divisors are equal to *one another* is not enough; they must equal the global order $L = \text{ord}_N(b)$. For example,

$$N = 9, \quad b = 2,$$

has the single prime divisor 3, with $\text{ord}_3(2) = 2$, but $\text{ord}_9(2) = 6$, so 9 is not total Midy to base 2.

5.5 A q -Midy refinement of the Pocklington mechanism

Midy's property can also force only a *part* of a prime power in $N - 1$ into every prime divisor of N . This is valuable precisely when the usual Pocklington gcd for the full prime power fails.

Proposition 5.5 (q -Midy congruence forcing; Castillo–García–Pulgarín–Velásquez–Soto). *Let $N > 1$ be odd, $\gcd(N, b) = 1$, and write*

$$N - 1 = q^s t, \quad q \text{ prime}, \quad q \nmid t.$$

If, for some $0 \leq i < s$,

$$N \mid \Phi_q(b^{q^{it}}), \tag{7}$$

then every prime divisor $p \mid N$ satisfies

$$p \equiv 1 \pmod{q^{i+1}}.$$

In particular, if

$$2(i+1) > s + \log_q t, \tag{8}$$

then N is prime.

Proof. This is Theorem 3.10 of [9]; the short order argument is worth recording. Put $x = b^{q^i t}$. If $p \mid N$, then $p \neq q$ because $q \mid N - 1$. From $p \mid \Phi_q(x)$ one has $x \not\equiv 1 \pmod{p}$ (otherwise $p \mid q$), while $x^q \equiv 1 \pmod{p}$. Hence x has exact order q modulo p . If $m = \text{ord}_p(b)$, then $m \mid q^{i+1}t$ because $x^q = 1$, but $m \nmid q^i t$ because $x \not\equiv 1$. Since $q \nmid t$, this says precisely that $v_q(m) = i + 1$. Therefore $q^{i+1} \mid m \mid p - 1$.

If N were composite, it would have a prime divisor $p \leq \sqrt{N}$. The preceding congruence gives $q^{i+1} \leq p - 1 < \sqrt{N}$, hence $q^{2(i+1)} < N$. On the other hand, condition (8) gives $q^{2(i+1)} > q^s t = N - 1$; since both sides are integers this implies $q^{2(i+1)} \geq N$, a contradiction. $\square$

Corollary 5.6 (Accumulating forced congruence factors). *Suppose independent order or Midy arguments prove, for a collection of prime powers $q^{e_q} \mid N - 1$, that*

$$q^{e_q} \mid p - 1 \quad \text{for every prime } p \mid N.$$

If a baseline order L is already known to divide every $p - 1$, define

$$F_{\text{force}} := \text{lcm}\left(L, \prod_q q^{e_q}\right).$$

Then every prime divisor of N is 1 modulo F_{force} . Consequently

$$F_{\text{force}} \geq \sqrt{N} \implies N \text{ is prime.}$$

The square-root threshold is not the only useful one. The familiar BLS cube-root discriminant argument depends only on the conclusion that every prime factor is 1 modulo a common forced modulus.

Proposition 5.7 (Cube-root test for an accumulated forced modulus). *Assume that every prime divisor $p \mid N$ satisfies $p \equiv 1 \pmod{F}$, where $F \mid N - 1$, and suppose*

$$N^{1/3} < F < \sqrt{N}.$$

Write the base- F expansion

$$N = c_2 F^2 + c_1 F + 1, \quad 0 \leq c_1, c_2 < F.$$

If

$$c_1^2 - 4c_2$$

is not a square, then N is prime.

Proof. Suppose N were composite and let $p \leq \sqrt{N}$ be a prime divisor. Write

$$p = aF + 1, \quad N/p = bF + 1, \quad a, b \geq 1,$$

using the forced congruence for p and for the complementary divisor. Then

$$N = abF^2 + (a + b)F + 1.$$

Since $N < F^3$, one has $ab < F$. Also $a + b < F$: otherwise $ab \geq a + b - 1 \geq F - 1$, and hence

$$N \geq (F - 1)F^2 + F^2 + 1 = F^3 + 1,$$

a contradiction. Therefore the displayed expression is already the base- F expansion, so $c_2 = ab$ and $c_1 = a + b$. Consequently

$$c_1^2 - 4c_2 = (a - b)^2$$

is a square. The contrapositive proves the assertion. $\square$

This gives a useful way to combine several auxiliary bases: the original period supplies one forced modulus, and q -Midy tests can add prime-power information that the original base does not expose. Crossing $\sqrt{N}$ gives immediate primality; crossing $N^{1/3}$ may already finish the proof by Proposition 5.7.

5.6 The part of $n - 1$ immediately visible to $N - 1$ tests

Assume now that

$$n - 1 = dL, \quad L = \text{ord}_n(b).$$

For each prime power $q^e \parallel n - 1$, one has

$$v_q(L) = e - v_q(d).$$

Define the *spectrally exposed factor*

$$F_L := \prod_{\substack{q^e \parallel n-1 \\ q \nmid d}} q^e. \tag{9}$$

Thus F_L consists exactly of the full prime-power factors of $n - 1$ whose q -adic exponent is already present in L ; equivalently, F_L is the largest divisor of $n - 1$ supported on primes not dividing d .

For each prime $q \mid F_L$ put

$$g_q = \gcd\left(b^{(n-1)/q} - 1, n\right). \tag{10}$$

Since $v_q(L) = v_q(n - 1)$, the order L does not divide $(n - 1)/q$, so $g_q < n$. Therefore either $g_q = 1$, or g_q is immediately a proper factor of n . If all $g_q = 1$, the hypotheses of the $N - 1$ Pocklington theorem hold for the completely factored part F_L . This observation is the basis of the Pocklington and BLS entries in the final toolbox.

5.7 Why the exact period is substantial input

The implications collected in the final toolbox begin with the exact value

$$L = \text{ord}_n(b).$$

This is important logically and computationally: the toolbox is not a “formula” which turns an arbitrary integer n into a practical primality test. For a general pair (b, n) , finding L may be much harder than merely testing n for primality.

In a generic group, the classical baby-step-giant-step and Pollard-rho order algorithms use on the order of $L^{1/2}$ group operations. Sutherland’s generic-order algorithms improve this for many orders, sometimes to roughly $L^{1/3}$, but no classical polynomial-time algorithm in $\log n$ is known for the unrestricted problem [36]. If a factored multiple of L is already known, order computation becomes much faster: one removes prime factors successively by modular exponentiation. For the multiplicative group modulo an unfactored composite n , however, obtaining such an exponent normally brings one back to integer factorization.

The standard general-purpose classical route is therefore factorization assisted. The heuristic complexity of the general number field sieve is

$$\exp\left(\left((64/9)^{1/3} + o(1)\right) (\log n)^{1/3} (\log \log n)^{2/3}\right),$$

with faster constants for suitable special forms [27]. Conversely, knowledge of multiplicative orders of random units can be used to factor composites by the classical post-processing at the heart of Shor's reduction. Quantum order finding, and hence factoring, is polynomial-time in the input length [31]; this does not presently make exact order a routine classical input.

There is a useful conceptual asymmetry here. Primality itself is known to be in deterministic polynomial time [2], whereas exact order finding for an arbitrary unknown composite modulus has no comparable classical polynomial-time method. Thus one should not first compute L merely in order to feed it into this toolbox. The toolbox is most useful when L arises naturally from the problem.

5.8 Cyclotomic values: a natural source of known periods

Cyclotomic factor tables provide an important example in which the period is known before primality is decided.

Proposition 5.8 (Primitive cyclotomic factors are prime or overpseudoprime). *Let $a \geq 2$, $L \geq 1$, and let $N > 1$ divide $\Phi_L(a)$. Suppose that*

$$\gcd(N, aL) = 1.$$

Then every divisor $D > 1$ of N satisfies

$$\text{ord}_D(a) = L.$$

Consequently N is either prime or an overpseudoprime to base a . Moreover $L \mid N - 1$.

Proof. Let $p^e \parallel D$. Since $p \nmid aL$ and $p \mid \Phi_L(a)$, the order of a modulo p is exactly L . Because $p^e \mid \Phi_L(a)$, one also has $a^L \equiv 1 \pmod{p^e}$. The order modulo p^e is therefore both a multiple of the order modulo p and a divisor of L , hence equals L . Taking least common multiples over the prime-power factors of D gives $\text{ord}_D(a) = L$. In particular every prime divisor p of N satisfies $p \equiv 1 \pmod{L}$, and hence $N \equiv 1 \pmod{L}$. $\square$

This proposition explains both the usefulness and the limitation of exact period information in tables such as Kamada's factorizations of $\Phi_L(10)$ [16]. Once exceptional factors dividing $10L$ are removed, an unresolved cofactor is already known to have exactly the same order L on every nontrivial divisor. Fermat, Midy, strong pseudoprimality and overpseudoprimality therefore cannot distinguish a prime cofactor from a composite one.

Within the present toolbox, the positive primality mechanism that can still use L is the $N - 1$ machinery. If $N - 1 = dL$ and F_L is the spectrally exposed factor of (9), then for primitive cyclotomic factors the Pocklington gcd conditions are automatic: for $q \mid F_L$ the exponent $(N - 1)/q$ is not divisible by L , so it is nontrivial modulo every prime divisor of N . Thus

$$F_L \geq \sqrt{N}$$

gives a Pocklington certificate, while the BLS cube-root regime may apply when $F_L^3 > N$ together with the usual nonsquare-discriminant condition. Since $F_L \leq L$, these conditions already show why only relatively small factors of a large cyclotomic value can usually be certified from the period alone.

Example 5.9 (One line of Kamada's base-10 table). *Kamada's entry for $L = 71401$ begins*

$$\Phi_{71401}(10) = 856813 \cdot 195152311403370773 \cdot C,$$

where C is an unresolved cofactor with 64877 decimal digits. Here

$$71401 = 11 \cdot 6491.$$

For the factor $N = 856813$,

$$N - 1 = 12 \cdot 71401, \quad F_L = 71401,$$

and

$$71401^2 = 5\,098\,102\,801 > N.$$

Thus the known order 71401 supplies a Pocklington proof of primality using base 10. By contrast,

$$195152311403370773 > 71401^3 = 364\,009\,638\,094\,201,$$

so even the clean BLS cube-root threshold cannot be reached using only the known period. The enormous cofactor is farther outside the range still. This is typical: the order information is perfect for recognizing the cyclotomic structure but, except for small factors, does not by itself settle primality.

5.9 Beyond the period-only threshold: a qualification and proof ladder

Suppose now that N is a probable-prime factor of a primitive cyclotomic value $\Phi_L(a)$, with $\gcd(N, aL) = 1$, but that N is too large for the factor L (or F_L) by itself to reach the Pocklington or BLS thresholds. The first conclusion should be negative but precise: *no further test that only rechecks the order of the same base a can distinguish prime from composite*. If N is composite, Proposition 5.8 already forces every nontrivial divisor to have that same order. Thus a Fermat test, a strong test, or total Midy to the cyclotomic base is already built into the hypothesis.

The period-only threshold is nevertheless not a ceiling for the rest of the toolbox. There are four natural next steps.

1. Enlarge the $N - 1$ certificate with auxiliary bases. Write $N - 1 = dL$. The factor F_L is only the portion of $N - 1$ automatically certified by the original base. It is not the largest factor that Pocklington can use. If $q^e \parallel N - 1$ is found by factoring an additional part of d , one may search for an auxiliary base u satisfying

$$u^{N-1} \equiv 1 \pmod{N}, \quad \gcd(u^{(N-1)/q} - 1, N) = 1.$$

Then every prime $p \mid N$ has $q^e \mid p - 1$. Different primes q may use different witnesses u . Accumulating these prime powers enlarges the certified factor of $N - 1$; if it reaches $\sqrt{N}$, Pocklington finishes the proof, and in the cube-root range the BLS criteria may finish it. If one of the gcds lies strictly between 1 and N , the same computation has instead found a proper factor.

2. Use partial q -adic forcing when full Pocklington fails. Proposition 5.5 is useful when an auxiliary base does not expose the full $q^e \parallel N - 1$. A congruence

$$N \mid \Phi_q(u^{q^{it}}), \quad N - 1 = q^s t,$$

still forces $q^{i+1} \mid p - 1$ for every $p \mid N$. Such partial powers can be combined through Corollary 5.6; once the accumulated modulus exceeds $N^{1/3}$, Proposition 5.7 may already finish the proof. For

the original cyclotomic base a this generally recovers only the q -part already present in L ; genuine amplification therefore requires auxiliary bases or additional information.

3. Do not restrict BLS to $N - 1$. The original Brillhart–Lehmer–Selfridge paper develops three families of criteria: tests using factors of $N - 1$, Lucas-sequence tests using factors of $N + 1$, and mixed tests using factors of both $N - 1$ and $N + 1$ [5]. Its stronger $N - 1$ results also use lower bounds for the prime factors of the still-unfactored remainder, so trial division or ECM progress can itself strengthen a certificate even before another factor is found. Thus a stubborn factorization of $N - 1$ is a reason both to exploit such remainder bounds and to inspect $N + 1$; the simplified cube-root entry in our toolbox is only one member of the full BLS family.

4. Separate qualification from proof. If the aim is first to increase confidence that a large unresolved factor is prime, use strong probable-prime tests to bases *independent* of the cyclotomic base. For an odd composite, at most one quarter of the admissible bases are strong Miller–Rabin liars, so k independent uniformly random strong tests have false-pass probability at most 4^{-k} with respect to the random choice of bases; this is a statement about the randomized test, not a Bayesian prior on N [11]. The original base a must not be counted among these independent tests: an overpseudoprime is forced to pass it. Strong Lucas/BPSW-type screening is another useful independent compositeness filter, but passing such filters is still not a proof.

When a proof rather than a qualification is required and the $N \pm 1$ machinery remains below threshold, one should switch to a general primality prover. APR–CL/Jacobi-sum methods descend from the algorithm of Adleman–Pomerance–Rumely [1, 10], while ECPP supplies compact elliptic-curve certificates [4]. These methods do not exploit the known reciprocal period directly; their role is precisely to finish cases in which the special structure has delivered all the information it can.

Kamada maintains a separate list of probable-prime factors whose primality has not yet been certified [17]. Such a “p” entry is different from an unresolved cofactor already known to be composite: the former is a candidate for the qualification/proof ladder above, whereas the latter needs factorization rather than a primality proof. As of August 2026 the current list begins with a 3747-digit probable-prime factor of $\Phi_{12198}(10)$. In such an example the five-digit period is enormously below the cube-root scale of the candidate, so the period-only certificate is not the realistic route; auxiliary $N \pm 1$ information or a general certificate method is needed.

6 Extended period–primality toolbox

This section is intentionally synthetic. It is meant to be read as a reference sheet rather than as a chain of proofs. Every small tool is accompanied by an elementary example. Most entries are classical; the point is to place them next to one another in the special notation

$$L = L_b(n) = \text{ord}_n(b).$$

Important warning. The toolbox assumes that the exact order L is already known. It is not a general-purpose primality test from n alone. Generic classical order computation is expensive, and for an unfactored composite modulus the factorization-assisted route can be harder than primality testing itself; see Subsection 5.7. The toolbox is most useful when L is supplied by the structure of the problem.

6.1 Fundamental order facts

Let $n > 1$, $b \geq 2$, and $\gcd(b, n) = 1$.

T1. Period equals order. The period of $1/n$ in base b is

$$L = \text{ord}_n(b), \quad L \mid \lambda(n) \mid \phi(n).$$

Example.

$$\frac{1}{7} = 0.\overline{142857}, \quad \text{ord}_7(10) = 6 = \lambda(7) = \phi(7).$$

T2. Every divisor of $\lambda(n)$ occurs. Because $(\mathbb{Z}/n\mathbb{Z})^\times$ is finite abelian, every divisor of its exponent $\lambda(n)$ is the order of some unit. In particular

$$\max_{\gcd(b,n)=1} L_b(n) = \lambda(n).$$

Example. For $n = 15$, $\lambda(15) = 4$. The divisors 1, 2, 4 occur as orders:

$$\text{ord}_{15}(16) = 1, \quad \text{ord}_{15}(14) = 2, \quad \text{ord}_{15}(2) = 4.$$

T3. Full totient order. For odd n , an equality $L = \phi(n)$ is possible exactly when $(\mathbb{Z}/n\mathbb{Z})^\times$ is cyclic; hence $n = p^k$ for an odd prime p . Thus $L = \phi(n)$ forces an odd prime power, but not necessarily a prime.

Example.

$$\phi(9) = 6, \quad \text{ord}_9(2) = 6,$$

yet 9 is composite.

T4. Spectral condition equals Fermat's congruence.

$$L \mid n - 1 \iff b^{n-1} \equiv 1 \pmod{n}.$$

For composite n this is exactly Fermat pseudoprimality to base b .

Example. For $n = 341 = 11 \cdot 31$ and $b = 2$,

$$L = \text{ord}_{341}(2) = 10 \mid 340,$$

so $2^{340} \equiv 1 \pmod{341}$. Carmichael numbers are the extreme all-base version of Fermat pseudoprimality; their infinitude was proved by Alford, Granville and Pomerance [3].

T5. Somer reduction under the spectral condition. If $L \mid n - 1$, then

$$L \mid \lambda_0(n) = \text{lcm}_{p \mid n}(p - 1).$$

Prime-power factors of n therefore do not supply extra factors to a spectral period.

Example. For $n = 25$, $b = 7$,

$$L = \text{ord}_{25}(7) = 4, \quad \lambda_0(25) = 5 - 1 = 4,$$

even though $\lambda(25) = 20$.

6.2 Direct primality and pseudoprimality consequences

T6. Lucas criterion. For odd $n > 1$,

$$\exists b : \text{ord}_n(b) = n - 1 \iff n \text{ is prime.}$$

Example.

$$\text{ord}_7(3) = 6 = 7 - 1,$$

so Lucas' criterion certifies 7. Recursive primality certificates based on a primitive-root witness and the prime factors of $p - 1$ lead to the classical Pratt certificate framework [28].

T7. Small spectral index. Somer proved fixed-index finiteness. The present paper adds explicit size obstructions and exact low-frequency lines.

Example. At $d = 4$ the complete spectral line is

$$\mathcal{E}(4) = \{9\}.$$

Thus any odd $n > 9$ satisfying

$$\text{ord}_n(b) = \frac{n-1}{4}$$

is prime.

T8. Repeated prime powers. In the notation

$$\kappa = \lambda(n)/L, \quad K = n/\text{rad}(n),$$

one has

$$K \mid \kappa.$$

Thus a prime divisor $p > \kappa$ occurs only to the first power.

Example. For the exceptional point $n = 25$, $d = 6$, $b = 7$,

$$L = 4, \quad \kappa = \frac{20}{4} = 5, \quad K = \frac{25}{5} = 5,$$

and indeed $K \mid \kappa$.

T9. Number of distinct prime factors. If $s = \omega(n)$ and n is an odd composite spectral point, then

$$2^{s-1}\kappa \leq d-1, \quad s \leq 1 + \left\lfloor \log_2 \frac{d-1}{\kappa} \right\rfloor.$$

Example. For $n = 561$, one may take $b = 5$ and $d = 7$:

$$L = 80, \quad \lambda(561) = 80, \quad \kappa = 1, \quad s = 3.$$

The inequality reads

$$2^{3-1} = 4 \leq 6 = d-1.$$

6.3 Midy's property from the exact period

Let $r > 1$ divide L and $h = L/r$.

T10. Exact block criterion. The sum of the r equal-length repetend blocks is divisible by $b^h - 1$ exactly when

$$n \mid 1 + b^h + \dots + b^{(r-1)h}.$$

For prime $r = q$ this is

$$n \mid \Phi_q(b^h).$$

Example. For $1/7$ in base 10, take $r = 2$, $h = 3$:

$$1 + 10^3 = 1001 = 7 \cdot 143.$$

Correspondingly,

$$142 + 857 = 999 = 10^3 - 1.$$

T11. Prime Midy property. If $n = p$ is prime, the Midy condition holds for every divisor $r > 1$ of L .

Example. Again $1/7 = 0.\overline{142857}$. Taking $r = 3$, $h = 2$ gives

$$14 + 28 + 57 = 99 = 10^2 - 1.$$

T12. The nines property. If $2 \mid L$ and $h = L/2$, the two halves add to $b^h - 1$ precisely when

$$b^h \equiv -1 \pmod{n}.$$

Example. For $n = 7$, $b = 10$,

$$10^3 \equiv -1 \pmod{7}$$

and the two halves 142 and 857 add to 999.

T13. Two-block Midy implies strong pseudoprimality in the spectral case. If n is composite, $L \mid n - 1$, and

$$b^{L/2} \equiv -1 \pmod{n},$$

then n passes the Miller–Rabin strong pseudoprime test to base b .

Example. For

$$n = 3277, \quad b = 2,$$

one has

$$L = \text{ord}_{3277}(2) = 28, \quad 28 \mid 3276, \quad 2^{14} \equiv -1 \pmod{3277}.$$

Thus 3277 is a strong pseudoprime to base 2.

T14. Total Midy needs only prime checks. The following conditions are equivalent:

Midy holds for every $r > 1$ dividing L ,

Midy holds for every prime $q \mid L$,

$\text{ord}_p(b) = L$ for every prime $p \mid n$,

$\text{ord}_D(b) = L$ for every divisor $D > 1$ of n .

Thus neither all block divisors of L nor all divisors of n have to be checked; see Proposition 5.4 and [9]. A composite satisfying these conditions is an overpseudoprime to base b and is automatically a strong pseudoprime to the same base [30, 8].

Example.

$$2047 = 23 \cdot 89, \quad \text{ord}_{23}(2) = \text{ord}_{89}(2) = \text{ord}_{2047}(2) = 11.$$

The two prime checks already prove that 2047 is total Midy, hence an overpseudoprime, to base 2.

T15. q -Midy forcing can refine Pocklington. If

$$n - 1 = q^s t, \quad q \nmid t,$$

and for some $0 \leq i < s$

$$n \mid \Phi_q(b^{q^i t}),$$

then every prime divisor $p \mid n$ satisfies

$$p \equiv 1 \pmod{q^{i+1}}.$$

If $2(i+1) > s + \log_q t$, this alone proves that n is prime; see Proposition 5.5 and [9].

Example. For $n = 1459$, $n - 1 = 3^6 \cdot 2$. Castillo, García-Pulgarín and Velásquez-Soto verify the criterion with $b = 2$, $q = 3$, and $i = 4$; hence every prime factor would be $1 \pmod{3^5}$, which is already too large for a proper factor not exceeding $\sqrt{1459}$. Thus 1459 is prime.

6.4 $N - 1$ primality certificates extracted from L

Suppose $L \mid n - 1$, $n - 1 = dL$, and let F_L be the spectrally exposed factor of (9).

T16. Automatic compositeness witness. For each prime $q \mid F_L$ put

$$g_q = \gcd\left(b^{(n-1)/q} - 1, n\right).$$

One always has $g_q < n$; if $g_q > 1$, it is a proper factor of n .

Example. For

$$n = 15, \quad b = 4, \quad L = 2, \quad d = 7,$$

the prime $q = 2$ is spectrally exposed and

$$\gcd(4^7 - 1, 15) = 3.$$

The order information itself produces the factor 3.

T17. Pocklington regime. If $g_q = 1$ for every $q \mid F_L$, every prime divisor p of n satisfies $p \equiv 1 \pmod{F_L}$. Hence

$$F_L \geq \sqrt{n} \implies n \text{ is prime.}$$

Example. For $n = 13$, $b = 2$,

$$L = 12, \quad F_L = 12 > \sqrt{13}.$$

The two required checks are

$$\gcd(2^6 - 1, 13) = 1, \quad \gcd(2^4 - 1, 13) = 1,$$

so Pocklington certifies 13.

T18. Certificate amplification beyond the original period. The exposed factor F_L is only what the known base certifies automatically. If $q^e \parallel n - 1$ lies in the remaining cofactor $d = (n - 1)/L$, an auxiliary base u satisfying

$$u^{n-1} \equiv 1 \pmod{n}, \quad \gcd\left(u^{(n-1)/q} - 1, n\right) = 1$$

forces $q^e \mid p - 1$ for every prime $p \mid n$. Different q may use different bases. Such prime powers may therefore be adjoined to the certified $N - 1$ factor until Pocklington or BLS reaches its threshold. A nontrivial gcd instead proves compositeness immediately.

Example. If the original order exposes $F_L = 2^a$ but factoring $(n - 1)/L$ also reveals q^e , a successful auxiliary witness for q upgrades the usable certified factor from 2^a to $2^a q^e$; the auxiliary base need not be the base whose reciprocal period supplied L .

T19. BLS cube-root regime. If

$$n^{1/3} < F_L < \sqrt{n}$$

and the Pocklington congruences hold, write

$$n = c_2 F_L^2 + c_1 F_L + 1, \quad 0 \leq c_1, c_2 < F_L.$$

If

$$c_1^2 - 4c_2$$

is not a square, the Brillhart–Lehmer–Selfridge cube-root criterion proves primality [5]. We use only this precise version here; stronger BLS variants have additional hypotheses and are not folded into the toolbox statement. More generally, F_L may be replaced by any accumulated forced modulus F for which every prime divisor of n is known to be 1 (mod F); see Proposition 5.7.

Example. Take $n = 71$. The element $b = 20$ has

$$\text{ord}_{71}(20) = 7, \quad 71 - 1 = 10 \cdot 7,$$

so $F_L = 7$. Since

$$71 = 1 \cdot 7^2 + 3 \cdot 7 + 1$$

and

$$3^2 - 4 \cdot 1 = 5$$

is not a square, the BLS cube-root criterion certifies 71.

T20. The full BLS toolbox also uses $N + 1$. The preceding entry is only the convenient $N - 1$ cube-root form. The original BLS theory also contains Lucas-sequence criteria based on factors of $N + 1$ and stronger mixed criteria using factors of both $N - 1$ and $N + 1$ [5]. When the known period leaves the $N - 1$ certificate too small, factoring $N + 1$ is therefore a natural second route.

Example. For a large cyclotomic probable prime with $L^3 < N$, failure of the period-only cube-root test says nothing about the factorization of $N + 1$; a large certified factor there can activate a Lucas or mixed BLS criterion.

T21. Primitive cyclotomic factors. If

$$N \mid \Phi_L(b), \quad \gcd(N, bL) = 1,$$

then every nontrivial divisor of N has order L . Thus N is either prime or overpseudoprime. All order-only tests in this toolbox therefore see the two cases in exactly the same way; the $N - 1$ certificate is the tool that can still turn the known L into a positive primality proof.

Example. Kamada lists

$$856813 \mid \Phi_{71401}(10).$$

Here

$$856813 - 1 = 12 \cdot 71401, \quad 71401^2 > 856813.$$

The primitive cyclotomic structure supplies the Pocklington gcd conditions, so the known period proves that 856813 is prime. On the same Kamada line, the 18-digit factor 195152311403370773 is already larger than 71401^3 , so the period alone does not reach even the clean BLS cube-root threshold.

T22. Independent bases qualify a cyclotomic PRP. An overpseudoprime is forced to pass the strong test to its defining cyclotomic base, so that base supplies no statistical evidence. Strong Miller–Rabin tests to independently chosen bases do: an odd composite has at most one quarter strong-liar bases, giving a false-pass probability at most 4^{-k} for k independent uniformly random choices. This qualifies a probable prime; it does not prove one [11].

Example. For a Kamada factor of $\Phi_L(10)$, base 10 should not be counted among the independent Miller–Rabin bases, because the cyclotomic structure already forces the required order behavior.

T23. General proof fallback: APR–CL or ECPP. If certificate amplification, q -Midy forcing, and the $N \pm 1$ BLS routes remain below threshold, a general primality prover is the correct final tool. APR–CL uses Jacobi sums, while ECPP recursively constructs elliptic-curve primality certificates [1, 10, 4]. These methods do not need the reciprocal period; they finish precisely the cases in which the special period structure no longer suffices.

Example. Kamada maintains a separate list of large probable-prime factors whose primality has not yet been certified [17]; those entries are natural inputs for a general certificate generator after the inexpensive structural and probable-prime filters have been exhausted.

6.5 Spectral distribution and related spectra

T24. Existential prime spectrum. Allowing the base to vary,

$$\mathcal{P}(d) = \{p \text{ prime} : p \equiv 1 \pmod{d}\}.$$

Each line is infinite and has relative density $1/\phi(d)$ among primes.

Example. For $d = 5$, the prime 31 belongs to the line because

$$31 \equiv 1 \pmod{5}.$$

Indeed,

$$\text{ord}_{31}(6) = 6 = \frac{31 - 1}{5}.$$

T25. Fixed-base prime spectrum. For one fixed base b , the set

$$\{p : \text{ord}_p(b) = (p - 1)/d\}$$

is the near-primitive-root problem. Under GRH it has the Wagstaff–Moree density described in Section 12.

Example. For the decimal base,

$$\text{ord}_{13}(10) = 6 = \frac{13 - 1}{2},$$

so 13 lies on the fixed-base-10 spectral line $d = 2$.

T26. λ - and ϕ -spectral hierarchy. Define

$$d_f(n) = \frac{n - 1}{\gcd(n - 1, f(n))}, \quad f \in \{\lambda, \phi\}.$$

Then

$$d_\phi(n) \mid d_\lambda(n),$$

and $d_\lambda(n)$ is the unique genuine L -spectral index of n .

Example. For $n = 21$,

$$\lambda(21) = 6, \quad \phi(21) = 12, \quad n - 1 = 20.$$

Hence

$$d_\lambda(21) = \frac{20}{2} = 10, \quad d_\phi(21) = \frac{20}{4} = 5,$$

so indeed $d_\phi(21) \mid d_\lambda(21)$.

6.6 One-page decision map

For quick use:

$L = n - 1$	$\implies$	n prime,
$L \mid n - 1$	$\iff$	$b^{n-1} \equiv 1 \pmod{n}$,
$L \mid n - 1$, n composite	$\implies$	Fermat pseudoprime to b ,
$2 \mid L$, two-block Midy	$\implies$	strong pseudoprime to b ,
$\text{ord}_p(b) = L \ \forall p \mid n$	$\iff$	total Midy / overpseudoprime,
$N \mid \Phi_q(b^{q^i t})$	$\implies$	$q^{i+1} \mid (p-1) \ \forall p \mid N$,
$F_L \geq \sqrt{n}$, $g_q = 1 \ \forall q \mid F_L$	$\implies$	n prime,
$F_{\text{force}} \geq \sqrt{N}$	$\implies$	N prime,
$F_{\text{force}} > N^{1/3}$, nonsquare digit discriminant	$\implies$	N prime,
$n \geq U_0(d)$, $L = (n-1)/d$	$\implies$	n prime.

The conceptual message is twofold. First, many classical statements become especially transparent once the exact reciprocal period L is placed at the center. Second, none of this removes the computational cost of obtaining L . The toolbox is therefore a structural reference and a conditional certificate generator, not a universal replacement for modern primality testing.

7 Structural bounds for composite spectral lines

Assume throughout this section that n is odd and composite and

$$L = L_b(n) = \frac{n-1}{d}.$$

Define

$$\kappa = \frac{\lambda(n)}{L}, \quad H = \frac{\phi(n)}{\lambda(n)}, \quad j = \frac{\phi(n)}{L} = \kappa H. \quad (11)$$

Then

$$d\lambda(n) = \kappa(n-1), \quad d\phi(n) = j(n-1).$$

Since $\phi(n) < n-1$,

$$1 \leq \kappa \leq j \leq d-1. \quad (12)$$

Write

$$n = \prod_{i=1}^s p_i^{a_i}, \quad 3 \leq p_1 < \dots < p_s, \quad s = \omega(n),$$

and put

$$K = \prod_{i=1}^s p_i^{a_i-1}.$$

Lemma 7.1 (Repeated-prime restriction). *One has*

$$K \mid \kappa.$$

Consequently, if $a_i > 1$, then

$$p_i^{a_i-1} \mid \kappa.$$

In particular every prime $p_i > \kappa$ occurs to the first power.

Proof. For odd prime powers,

$$\lambda(p_i^{a_i}) = p_i^{a_i-1}(p_i - 1).$$

Hence $K \mid \lambda(n)$. From $d\lambda(n) = \kappa(n - 1)$ we get $K \mid \kappa(n - 1)$. But every prime divisor of K divides n , and hence $\gcd(K, n - 1) = 1$. Therefore $K \mid \kappa$. $\square$

Lemma 7.2 (Two-adic overlap). *For odd n with s distinct prime divisors,*

$$2^{s-1} \mid H = \frac{\phi(n)}{\lambda(n)}.$$

Consequently

$$2^{s-1}\kappa \mid j, \quad 2^{s-1}\kappa \leq d - 1. \quad (13)$$

Proof. If $e_i = v_2(p_i - 1)$, then

$$v_2(\phi(n)) = \sum_i e_i, \quad v_2(\lambda(n)) = \max_i e_i.$$

Since all $e_i \geq 1$,

$$v_2(H) = \sum_i e_i - \max_i e_i \geq s - 1.$$

The second assertion follows from $j = \kappa H \leq d - 1$. $\square$

Corollary 7.3 (Number of distinct prime factors). *Every composite on spectral line d satisfies*

$$s \leq 1 + \left\lfloor \log_2 \left(\frac{d-1}{\kappa} \right) \right\rfloor \leq 1 + \lfloor \log_2(d-1) \rfloor.$$

Proposition 7.4 (Smallest-prime bound). *One has*

$$p_1 < \frac{d + j(s-1)}{d - j},$$

and therefore

$$p_1 \leq s(d-1).$$

Proof. From $d\phi(n) = j(n-1)$,

$$\frac{d}{j} = \frac{n-1}{\phi(n)} < \frac{n}{\phi(n)} = \prod_{i=1}^s \frac{p_i}{p_i-1}.$$

Since the primes are distinct and odd, $p_i \geq p_1 + 2(i-1)$, and hence a fortiori $p_i \geq p_1 + i - 1$,

$$\prod_{i=1}^s \frac{p_i}{p_i-1} \leq \frac{p_1 + s - 1}{p_1 - 1}.$$

Solving for p_1 gives the result. $\square$

Lemma 7.5 (A uniform Euler-factor ratio bound). *If $3 \leq p_1 < \dots < p_r$ are distinct odd primes, then*

$$\prod_{i=1}^r \frac{p_i}{p_i-1} \leq \prod_{i=1}^r \frac{i+2}{i+1} = \frac{r+2}{2} < r+1.$$

Proof. Since $p_i \geq i+2$ and $x/(x-1)$ is decreasing for $x > 1$, each factor is at most $(i+2)/(i+1)$; the latter product telescopes. $\square$

For

$$P_k = \prod_{i=1}^k p_i^{a_i},$$

the following recursive restriction is useful.

Proposition 7.6 (Squarefree-suffix bound). *If $2 \leq k \leq s$ and $p_k > \kappa$, then $p_k, \dots, p_s$ occur only to the first power and*

$$p_k < d\lambda(P_{k-1})(s - k + 1).$$

Proof. By Lemma 7.1, the suffix

$$q := \prod_{i=k}^s p_i$$

is squarefree. Put $P = P_{k-1}$ and

$$J_k := \frac{\lambda(P)\phi(q)}{\lambda(n)}.$$

Since $\lambda(n) = \text{lcm}(\lambda(P), \lambda(q))$ and $\lambda(q) \mid \phi(q)$, the number J_k is an integer. Multiplying

$$d\lambda(n) = \kappa(n - 1)$$

by J_k gives

$$d\lambda(P)\phi(q) = \kappa J_k(Pq - 1).$$

Write

$$\phi(q) = q - \Delta_q.$$

Then

$$q(d\lambda(P) - \kappa J_k P) = d\lambda(P)\Delta_q - \kappa J_k.$$

The coefficient on the left is a positive integer, because

$$d\lambda(P) - \kappa J_k P = d\lambda(P) \frac{P(q - \phi(q)) - 1}{Pq - 1} > 0.$$

Hence it is at least 1, and therefore

$$q < d\lambda(P)\Delta_q.$$

Finally,

$$\frac{\Delta_q}{q} = 1 - \prod_{i=k}^s \left(1 - \frac{1}{p_i}\right) < \sum_{i=k}^s \frac{1}{p_i} \leq \frac{s - k + 1}{p_k}.$$

Combining the last two inequalities proves

$$p_k < d\lambda(P_{k-1})(s - k + 1).$$

□

Proposition 7.7 (Largest-prime bound). *For $s \geq 2$,*

$$p_s \leq \kappa(P_{s-1} - 1) + 1.$$

More precisely, if $p_s > \kappa$, then

$$p_s - 1 \mid \kappa(P_{s-1} - 1).$$

Proof. If $p_s \leq \kappa$, then the displayed upper bound is immediate because $P_{s-1} \geq 3$. Suppose therefore that $p_s > \kappa$. Lemma 7.1 gives $a_s = 1$. Since $p_s - 1 \mid \lambda(n)$ and $d\lambda(n) = \kappa(n - 1)$,

$$p_s - 1 \mid \kappa(n - 1).$$

Writing $n = P_{s-1}p_s$ and reducing modulo $p_s - 1$ gives

$$p_s - 1 \mid \kappa(P_{s-1} - 1),$$

and hence $p_s \leq \kappa(P_{s-1} - 1) + 1$.

□

7.1 A uniform spectral size obstruction

Let

$$L = \frac{n-1}{d}, \quad \kappa = \frac{\lambda(n)}{L}, \quad H = \frac{\phi(n)}{\lambda(n)}, \quad j = \kappa H.$$

For an odd composite n with $s = \omega(n)$ distinct prime factors,

$$2^{s-1} \mid H, \quad j \leq d-1,$$

and hence

$$2^{s-1} \kappa \leq d-1. \tag{U1}$$

For $d \geq 2$, define

$$S = S(d) := 1 + \lfloor \log_2(d-1) \rfloor, \quad M = M(d) := 2^{S-1}.$$

Thus M is the largest power of 2 not exceeding $d-1$.

Theorem 7.8 (Uniform spectral size obstruction). *Suppose that $n > 1$ is odd and*

$$L_b(n) = \frac{n-1}{d}$$

for some $b \geq 2$ coprime to n . If n is composite, then

$$\boxed{n < U_0(d) := (dS(d))^{2M(d)}}. \tag{U2}$$

Consequently

$$n \geq U_0(d) \implies n \text{ is prime.}$$

Here $d \geq 2$; for $d = 1$ the conclusion is Lucas's criterion.

Proof. The case $d = 1$ is immediate from $L_b(n) = n-1$, so assume $d \geq 2$. Write

$$n = K \prod_{i=1}^s p_i, \quad K = \prod_{i=1}^s p_i^{a_i-1}, \quad 3 \leq p_1 < \dots < p_s.$$

The repeated-prime restriction gives $K \mid \kappa$. In particular,

$$n \leq \kappa R_s, \quad R_k := \prod_{i=1}^k p_i, \tag{U3}$$

and $s \leq S$ by (U1).

We first dispose of the prime-power case $s = 1$, which is not covered by the terminal-prime estimate below. Then $n = p^a$ with $a \geq 2$ and $K = p^{a-1} \mid \kappa \leq d-1$. Since $p \leq p^{a-1} = K$,

$$n = pK \leq K^2 \leq (d-1)^2 < d^2 \leq U_0(d).$$

Henceforth assume $s \geq 2$.

For $2 \leq k \leq s-1$, put

$$A_{k-1} := \prod_{i < k} (p_i - 1), \quad K_{k-1} := \prod_{i < k} p_i^{a_i-1}.$$

Then $P_{k-1} = K_{k-1} \prod_{i < k} p_i$ has $k-1$ distinct odd prime factors and $K_{k-1} \mid K \mid \kappa$. Therefore

$$\lambda(P_{k-1}) \leq \frac{\phi(P_{k-1})}{2^{k-2}} = \frac{K_{k-1} A_{k-1}}{2^{k-2}} \leq \frac{\kappa A_{k-1}}{2^{k-2}}.$$

The squarefree-suffix estimate yields

$$p_k < \frac{d\kappa(s-k+1)}{2^{k-2}} A_{k-1} \quad \text{whenever } p_k > \kappa. \quad (\text{U4})$$

If $p_k \leq \kappa$, the weaker inequality

$$p_k < d\kappa(s-k+1)A_{k-1}$$

is automatic. Thus, for all $2 \leq k \leq s-1$,

$$A_k = A_{k-1}(p_k - 1) < d\kappa S A_{k-1}^2. \quad (\text{U5})$$

We distinguish two cases.

Case 1: $s = S$. Since

$$M = 2^{S-1} \leq d-1 < 2M,$$

(U1) forces $\kappa = 1$. Moreover H is a positive multiple of M and $j = H \leq d-1 < 2M$, hence

$$H = j = M.$$

The smallest-prime bound gives

$$p_1 < \frac{d+M(S-1)}{d-M} < dS,$$

so $A_1 < dS$. Iterating (U5) gives

$$A_{S-1} < (dS)^{M-1}. \quad (\text{U6})$$

By Lemma 7.5,

$$\frac{R_{S-1}}{A_{S-1}} = \prod_{i=1}^{S-1} \frac{p_i}{p_i - 1} \leq \frac{S+1}{2} < S,$$

and therefore

$$R_{S-1} < S(dS)^{M-1}.$$

Here $\kappa = K = 1$, so n is squarefree. The terminal-prime divisibility is

$$p_S - 1 \mid R_{S-1} - 1.$$

Because $S \geq 2$, one has $R_{S-1} - 1 > 0$, hence $p_S \leq R_{S-1}$. Consequently

$$n = R_{S-1}p_S \leq R_{S-1}^2 < S^2(dS)^{2M-2} < (dS)^{2M}.$$

Case 2: $2 \leq s < S$. Put $M_s = 2^{s-1}$, so $M_s \leq M/2$. Using only $\kappa < d$, $s \leq S$, and $p_1 < dS$, equation (U5) gives

$$A_{s-1} < (d^2 S)^{M_s-1}.$$

Lemma 7.5 gives

$$R_{s-1} < S(d^2 S)^{M_s-1}. \quad (\text{U7})$$

The terminal-prime bound and $P_{s-1} \leq \kappa R_{s-1}$ imply

$$p_s \leq \kappa(P_{s-1} - 1) + 1 \leq \kappa P_{s-1} \leq \kappa^2 R_{s-1}.$$

Using (U3), $\kappa < d$, (U7), and $2M_s \leq M$,

$$n < d^3 R_{s-1}^2 < d^3 S^2 (d^2 S)^{2M_s-2} \leq d^{2M-1} S^M < (dS)^{2M}.$$

This completes the proof. □

Corollary 7.9 (Asymptotic primality criterion). *For composite spectral points,*

$$\log n \leq 2d(\log d + \log \log d + O(1)).$$

Equivalently,

$$d \geq \left(\frac{1}{2} + o(1)\right) \frac{\log n}{\log \log n}.$$

Thus for every fixed $\varepsilon > 0$, all sufficiently large n satisfying

$$d \leq \left(\frac{1}{2} - \varepsilon\right) \frac{\log n}{\log \log n}$$

are prime.

Proof. Since $M(d) \leq d - 1$ and $S(d) = O(\log d)$, Theorem 7.8 gives

$$\log U_0(d) \leq 2d(\log d + \log \log d + O(1)).$$

Fix $\varepsilon > 0$ and suppose that

$$d \leq \left(\frac{1}{2} - \varepsilon\right) \frac{\log n}{\log \log n}.$$

Only upper bounds on d are needed. Uniformly in this range,

$$\log d \leq \log \log n + O(\log \log \log n), \quad \log \log d = O(\log \log \log n)$$

whenever $d \rightarrow \infty$; bounded d are easier and satisfy the same final estimate. Therefore

$$\begin{aligned} \log U_0(d) &\leq 2d(\log d + \log \log d + O(1)) \\ &\leq (1 - 2\varepsilon) \frac{\log n}{\log \log n} (\log \log n + O(\log \log \log n)) \\ &= (1 - 2\varepsilon + o(1)) \log n < \log n \end{aligned}$$

for all sufficiently large n . Hence $n > U_0(d)$, so Theorem 7.8 forces n to be prime. This proves the stated criterion without assuming a lower bound on d . $\square$

Remark 7.10. *The coefficient $1/2$ is the worst leading coefficient produced by the uniform bound, rather than a claimed sharp asymptotic constant for actual composite spectral points. It is governed by the dyadic extremal values*

$$M(d) = d - 1 \iff d = 2^r + 1.$$

At $d = 2^r$ one has $M(d) = d/2$, and the corresponding local leading coefficient in this bound is 1 rather than $1/2$. Retaining the exact branch parameters (s, κ, H, j) gives much smaller finite bounds away from the dyadic spikes.

Remark 7.11 (Comparison with Somer's effective finiteness). *Somer's proof of fixed- d finiteness is effective in the sense that, for each prescribed d , it eventually produces a computable constant beyond which no Fermat d -pseudoprime can occur. He explicitly remarks, however, that his argument does not provide a uniform procedure for obtaining this constant as a function of arbitrary d [33]. The point of $U_0(d)$ is therefore not finiteness itself, which is Somer's theorem, but uniformity: it is one closed expression valid for every d . The asymptotic consequence*

$$d \geq \left(\frac{1}{2} + o(1)\right) \frac{\log n}{\log \log n}$$

for composite spectral points is a quantitative form of the statement that small indices are rare.

8 Somer's small-index theorem and a unified three-page proof

Somer's Theorem 3 [33] gives a complete table of his Fermat d -pseudoprimes for $1 \leq d \leq 8$. With his convention excluding bases $a \equiv \pm 1 \pmod{N}$, the table is

d	composite Fermat d -pseudoprimes
1, 2, 3, 4	$\emptyset$
5	6601
6	25
7	15, 561
8	49

The present spectral convention allows every integer base $b \geq 2$ coprime to n . It therefore adds the order-two occurrence $\text{ord}_9(8) = 2 = (9 - 1)/4$ and the order-one occurrence $\text{ord}_9(10) = 1 = (9 - 1)/8$.

Somer's published paper gives a detailed proof only for $d = 1$ and $d = 5$, describing the latter as illustrative; it refers the six remaining cases to his dissertation. We rely here on Somer's published statement of that reference rather than on an independent inspection of the dissertation pages. Since a complete proof of the small range is useful to readers and to later computational checks, we give here a single argument handling all $1 \leq d \leq 8$ at once. It is independent of Somer's case-by-case calculation and uses the structural lemmas of the preceding section.

Theorem 8.1 (Small-index classification). *Let $n > 1$ be odd, let $b \geq 2$ with $\gcd(b, n) = 1$, and suppose*

$$L_b(n) = \frac{n-1}{d}, \quad 1 \leq d \leq 8.$$

Then n is prime except for

d	composite exceptions
4	9
5	6601
6	25
7	15, 561
8	9, 49.

Every listed exception actually occurs; witnesses are

$$\text{ord}_9(8) = 2, \quad \text{ord}_{6601}(11) = 1320, \quad \text{ord}_{25}(7) = 4, \quad \text{ord}_{15}(4) = 2,$$

$$\text{ord}_{561}(5) = 80, \quad \text{ord}_9(10) = 1, \quad \text{ord}_{49}(19) = 6.$$

Proof. Put

$$\ell := L_b(n) = \frac{n-1}{d}, \quad \phi(n) = j\ell, \quad \lambda(n) = \kappa\ell.$$

Because $\lambda(n) \mid \phi(n)$,

$$\kappa \mid j.$$

For composite n , $\phi(n) < n - 1 = d\ell$, so

$$1 \leq \kappa \leq j \leq d - 1 \leq 7. \tag{S1}$$

Write

$$n = \prod_{i=1}^s p_i^{a_i}, \quad 3 \leq p_1 < \cdots < p_s, \quad K = \prod_i p_i^{a_i-1}.$$

By Lemma 7.1, $K \mid \kappa$. By Lemma 7.2,

$$2^{s-1}\kappa \mid j.$$

Together with (S1), this gives $s \leq 3$. We treat the three possibilities separately.

Case 1: one distinct prime factor. Let $n = p^a$, $a \geq 2$. Then $\lambda(n) = \phi(n)$, hence $\kappa = j$, and $d\phi(n) = j(n-1)$ becomes

$$p^{a-1}((d-j)p-d) = -j. \quad (\text{S2})$$

Thus $p^{a-1} \mid j \leq 7$. Hence $a = 2$ and $p \in \{3, 5, 7\}$. Substitution in (S2) gives only

$$(d, j, n) = (4, 3, 9), \quad (6, 5, 25), \quad (8, 6, 9), \quad (8, 7, 49).$$

These are exactly the prime-power exceptions displayed in the theorem.

Case 2: two distinct prime factors. Write

$$n = p^a q^c, \quad p < q, \quad P = p^{a-1} q^{c-1} = K.$$

The equation $d\phi(n) = j(n-1)$ is

$$P((d-j)pq - d(p+q) + d) = -j. \quad (\text{S3})$$

Here $2\kappa \mid j$ and $P \mid \kappa$. Since $j \leq 7$ and P is odd, a nonsquarefree solution would force $P = 3$, hence $n = 9q$ with $q > 3$ prime. Here $P = 3$ forces $\kappa = 3$, and $2\kappa \mid j \leq 7$ therefore gives $j = 6$. Equation (S3) reduces to

$$6d(q-1) = 6(9q-1), \quad q = \frac{d-1}{d-9} < 0$$

for $d \leq 8$. Thus no nonsquarefree two-prime solution occurs.

Putting $P = 1$ in (S3) and completing the product gives the useful factorization

$$((d-j)p-d)((d-j)q-d) = j^2. \quad (\text{S4})$$

For a valid solution the two factors in (S4) are positive: solving the master equation for the larger prime gives a positive denominator. The two-adic overlap budget should also be used before any further enumeration: since $2\kappa \mid j$, the integer j is even. This parity condition is what removes the formal odd- j master-equation solutions (such as 21, 33, 65 on other spectral lines). Together with $j \leq 7$, this leaves only $j \in \{2, 4, 6\}$. Factoring (S4) over these three small squares gives exactly

d	j	(p, q)	n	$\lambda(n)$	$\ell = (n-1)/d$
5	4	(7, 13)	91	12	18
7	4	(3, 5)	15	4	2
8	6	(5, 13)	65	12	8.

The defining order can exist only if $\ell \mid \lambda(n)$. This removes the first and third rows, leaving only $n = 15$ at $d = 7$, and indeed $\text{ord}_{15}(4) = 2$.

Case 3: three distinct prime factors. Now

$$4\kappa \mid j.$$

Since $j \leq 7$, necessarily

$$\kappa = 1, \quad j = 4. \quad (\text{S5})$$

Because $K \mid \kappa$, we immediately get $K = 1$: the integer is squarefree,

$$n = pqr, \quad p < q < r.$$

The Euler equation is

$$d(p-1)(q-1)(r-1) = j(pqr-1). \quad (\text{S6})$$

For fixed p , put

$$A = (d-j)p - d, \quad B = d(p-1).$$

Equation (S6) becomes

$$Aqr - B(q+r) + B + j = 0,$$

and hence

$$(Aq - B)(Ar - B) = j(d(p-1)^2 + jp). \quad (\text{S7})$$

Furthermore, (S6) implies

$$(d-j)pqr < d(pq + pr + qr) < 3dqr,$$

and a valid solution also requires $A = (d-j)p - d > 0$, hence $p > d/(d-j)$. Thus

$$\frac{d}{d-j} < p < \frac{3d}{d-j}. \quad (\text{S8})$$

Thus only a handful of primes p can occur, and for each one (S7) reduces the remaining search to divisor pairs of one explicit integer. Concretely, the bound gives $p = 7, 11, 13$ for $d = 5$, $p = 5, 7$ for $d = 6$, $p = 3, 5$ for $d = 7$, and $p = 3, 5$ for $d = 8$; checking the positive divisor pairs in (S7) leaves precisely the three prime triples displayed below (all other pairs contain a composite entry or violate the required ordering). Since $j = 4$ already, the complete candidate list has only three rows:

d	j	(p, q, r)	n	$\lambda(n)$	$\ell = (n-1)/d$
5	4	(7, 19, 67)	8911	198	1782
5	4	(7, 23, 41)	6601	1320	1320
7	4	(3, 11, 17)	561	80	80.

By (S5) one must have $\lambda(n) = \ell$. Exactly two rows survive:

$$6601 \quad (d=5), \quad 561 \quad (d=7).$$

Combining the three cases gives

d	$\mathcal{E}(d)$
1, 2, 3	$\emptyset$
4	$\{9\}$
5	$\{6601\}$
6	$\{25\}$
7	$\{15, 561\}$
8	$\{9, 49\}$.

The displayed witness bases verify every surviving entry. □

8.1 Comparison with Somer's proof

The two proofs emphasize different structures. Somer introduces $\lambda'(N) = \text{lcm}_{p|N}(p-1)$ and an auxiliary product $\lambda''(N)$, proves the factor-count bound $t \leq \lfloor \log_2 d + 1 \rfloor$, bounds repeated prime powers, and then handles each small value of d separately. In the 1989 paper only the $d = 5$ case is written out in detail: after several reductions it isolates $7 \cdot 23 \cdot 41 = 6601$.

Our proof packages the same small-index scarcity into the two integers

$$\kappa = \lambda(n)/L, \quad H = \phi(n)/\lambda(n),$$

with $j = \kappa H$. The exact divisibilities $K \mid \kappa$ and $2^{s-1}\kappa \mid j$ reduce the whole range at once to $s \leq 3$. The cases $s = 2$ and $s = 3$ are then controlled by the two explicit factorizations (S4) and (S7). The resulting proof is short enough to reproduce in full, while retaining a transparent candidate list that can be checked by hand.

The classification itself should therefore be credited to Somer for his nontrivial-base convention. What is new here in the small range is the unified proof, the inclusion of the two trivial-base spectral occurrences, and the integration of the result into the effective spectral framework used for larger d .

The later Carlip–Jacobson–Somer treatment [6] also recovers the general Fermat factor-count bound $t < \log_2 d + 1$ and controls prime powers. The present argument should therefore not be read as claiming novelty for those qualitative restrictions. Its sharper bookkeeping retains κ and H separately: $K \mid \kappa$ records the complete repeated-prime part, while $2^{s-1}\kappa \mid j \leq d - 1$ couples repetition and overlap. This is what makes the uniform branch bounds and the later counting argument possible.

9 Certified computer-assisted classification through $d = 64$

For larger d the number of possible distinct prime factors grows:

$$s \leq 4 \quad (9 \leq d \leq 16), \quad s \leq 5 \quad (17 \leq d \leq 32).$$

A hand proof by separate Diophantine branches becomes increasingly unattractive. We therefore use a certified exact enumeration of the finite factorization space.

Write

$$n = P \prod_{i=1}^s p_i, \quad P = \prod_{i=1}^s p_i^{a_i-1}.$$

Since

$$d\phi(n) = j(n-1), \quad P \mid j,$$

write

$$j = cP.$$

Then every candidate satisfies the master equation

$$d \prod_{i=1}^s (p_i - 1) = c \left(P \prod_{i=1}^s p_i - 1 \right). \quad (14)$$

The ranges

$$s \leq 1 + \lfloor \log_2(d-1) \rfloor, \quad 1 \leq P \leq d-1, \quad 1 \leq c \leq \left\lfloor \frac{d-1}{P} \right\rfloor$$

are finite.

Suppose the first $s-1$ primes have been chosen and put

$$A = \prod_{i < s} (p_i - 1), \quad B = \prod_{i < s} p_i.$$

Equation (14) is linear in the final prime:

$$p_s = \frac{dA - c}{dA - cPB}. \quad (15)$$

Thus the last prime is solved algebraically rather than scanned.

For counting purposes it is useful to stop one step earlier and solve the last *two* primes simultaneously. This gives an exact divisor problem rather than a second prime scan.

Lemma 9.1 (Divisor-pair completion of the last two primes). *Fix a master-equation branch (d, s, j, P) , put $c = j/P$, and suppose the first $r = s - 2$ primes have been chosen. Write*

$$A = \prod_{i \leq r} (p_i - 1), \quad B = \prod_{i \leq r} p_i,$$

with the empty products equal to 1. If the two remaining primes are $x < y$, define

$$a = dA, \quad b = cPB = jB, \quad \Delta = a - b.$$

Every valid completion has $\Delta > 0$ and satisfies

$$\boxed{(\Delta x - a)(\Delta y - a) = T, \quad T := a(b - c) + bc.} \quad (16)$$

Consequently, for a fixed branch and prefix, there are at most $\lfloor \tau(T)/2 \rfloor$ ordered prime completions.

Proof. The master equation becomes

$$a(x - 1)(y - 1) = bxy - c,$$

or

$$\Delta xy - a(x + y) + (a + c) = 0.$$

Solving for y gives

$$y = \frac{a(x - 1) - c}{\Delta x - a}.$$

The numerator is positive: $a = dA \geq d$, $x \geq 3$, and $c \leq j \leq d - 1$. Hence the denominator is positive for every valid completion, and therefore $\Delta > 0$. Multiplying the preceding bilinear equation by Δ and completing the product gives

$$(\Delta x - a)(\Delta y - a) = a^2 - \Delta(a + c) = a(b - c) + bc.$$

The two positive factors on the left form a divisor pair of T ; because $x < y$, at most half of the positive divisors can occur as the first factor. Integrality, primality, the support condition for P , and the final λ -divisibility checks can only reduce this number. $\square$

The principal computational application of this lemma is the difficult $(d, s, \kappa, H, P) = (49, 5, 3, 16, 1)$ branch in Subsection 9.2: after the first three primes are exhausted, exactly 49 496 prefixes remain and the last two primes are then completed entirely by (16).

At an intermediate prefix, put

$$A_r = \prod_{i \leq r} (p_i - 1), \quad B_r = \prod_{i \leq r} p_i, \quad R = \frac{dA_r}{cPB_r}.$$

If $R \leq 1$, the branch is impossible. Suppose that $R > 1$, that t primes remain, and that the next prime is x . Dividing (14) by cPB_r gives the necessary inequality

$$R \prod_{\text{remaining } q} \left(1 - \frac{1}{q}\right) < 1.$$

Every remaining prime satisfies $q \geq x$, so

$$\prod_{\text{remaining } q} \left(1 - \frac{1}{q}\right) \geq \left(\frac{x - 1}{x}\right)^t.$$

Hence every surviving branch must satisfy

$$\boxed{x^t cPB_r > (x-1)^t dA_r.} \quad (17)$$

The predicate is monotone in x , so the largest admissible x is obtained by binary search using exact integer comparisons. No floating-point approximation is allowed to decide whether a branch is discarded.

Every completed candidate is checked for

$$d \mid n-1, \quad \frac{n-1}{d} \mid \lambda(n).$$

The first of these checks is essential: an earlier exploratory script used integer division without the divisibility test and thereby created several false positives in preliminary tables.

The overlap pruning used below can be updated recursively. Put

$$c_i = p_i^{a_i-1}(p_i-1), \quad \Lambda_k = \text{lcm}(c_1, \dots, c_k), \quad H_k = \frac{\prod_{i=1}^k c_i}{\Lambda_k}.$$

Then

$$H_k = H_{k-1} \gcd(\Lambda_{k-1}, c_k). \quad (18)$$

Thus each new prime consumes part of the finite overlap budget $H_s = \phi(n)/\lambda(n)$. In the minimal-overlap case $H_s = 2^{s-1}$ this forces $\gcd(c_i, c_j) = 2$ for $i \neq j$; for squarefree n , $\gcd(p_i-1, p_j-1) = 2$.

Proposition 9.2 (Finite and complete search specification). *For fixed spectral index d , every composite solution occurs in the following finite search.*

- (1) *Handle $s = 1$ separately: enumerate odd prime powers p^a , $a \geq 2$, with $p^{a-1} \leq d-1$, and retain exactly those satisfying $d \mid p^a - 1$ and $(p^a - 1)/d \mid \lambda(p^a)$.*
- (2) *For $s \geq 2$, enumerate*

$$2 \leq s \leq S(d), \quad 1 \leq \kappa \leq \frac{d-1}{2^{s-1}}, \quad H \in \{2^{s-1}, 2 \cdot 2^{s-1}, \dots\}, \quad \kappa H \leq d-1.$$

For each branch set $j = \kappa H$, enumerate $P \mid \kappa$, and put $c = j/P$.

- (3) *Factor P . Every prime dividing $P = n/\text{rad}(n)$ is a required support prime and $v_q(P) = a_q - 1$. Recursion uses strictly increasing distinct primes.*
- (4) *Apply only necessary prunes: required-prime ordering and remaining-slot checks; $H_k \mid H$ and the remaining overlap budget from (18); $R > 1$; and the exact ratio bound (17).*
- (5) *Solve the final prime by (15). Reject it if it is not an integer prime strictly larger than the preceding prime, if required support is missing, or if the exact terminal overlap is not H . Finally recompute $n, \phi(n), \lambda(n), L$ and all defining divisibilities.*

For $s \geq 2$ the number of parameter branches before prefix recursion is

$$B(d) = \sum_{s=2}^{S(d)} \sum_{\kappa \leq (d-1)/2^{s-1}} \left\lfloor \frac{d-1}{2^{s-1}\kappa} \right\rfloor \tau(\kappa).$$

Proof. The prime-power restriction in step (1) follows from $p^{a-1} \mid \kappa \leq d-1$. For $s \geq 2$, Corollary 7.3, Lemma 7.2, and Lemma 7.1 place every genuine solution in one enumerated (s, κ, H, P) branch. The identity $P = \prod p_i^{a_i-1}$ proves the support rule.

No recursive prune can remove a prefix of a genuine solution. If $R \leq 1$, appending another prime multiplies R by $(q-1)/q < 1$, whereas positivity of the denominator in (15) requires $R > 1$ immediately before the last prime. The ratio bound is the necessary inequality derived above. For the overlap tests, H_k must divide the target H , and after the first local factor every future multiplier $\gcd(\Lambda_{i-1}, c_i)$ is at least 2 because both arguments are even; this proves the remaining-budget test. The final prime in (15) is the unique possible algebraic completion. Requiring it to exceed the prefix loses nothing because every factorization is enumerated in increasing order. Finally $H_s = H$ is an identity for the fixed branch, so the exact terminal check is necessary. The final arithmetic tests are exactly the conditions of Proposition 2.6. $\square$

The implementation follows the proposition literally. In pseudocode:

```

prime_power_solutions(d)
for s, kappa, H, P in all admissible branches:
    required := prime_support(P)
    recurse(increasing_prefix):
        reject if required-prime order/slots are impossible
        reject unless H_prefix divides H and overlap budget can finish
        reject if R <= 1
        bound next prime by the exact ratio inequality
        recurse over every prime within that exact bound
        at length s-1 solve the last prime algebraically
        reject unless last prime > prefix and required support is complete
        reject unless H_full == H
        recompute phi, lambda, L and all defining divisibilities

```

Thus the written specification includes every mathematical prune used by the reference search.

Theorem 9.3 (Computer-assisted classification for $9 \leq d \leq 32$). *For every $9 \leq d \leq 32$, $\mathcal{E}(d)$ is exactly the corresponding row of Table 2.*

Proof. Proposition 9.2 proves completeness of the finite search. The reference implementation exhausts every branch with exact integer arithmetic and an arithmetic verifier independently recomputes every reported solution. No integer-size cutoff is used in the Diophantine search. $\square$

9.1 The dyadic transition at $d = 33$

At $d = 33$, six distinct prime factors become structurally possible. The two largest $\kappa = 1$ branches can be removed analytically.

Lemma 9.4 (Odd-index 2-adic parity). *Let $n = \prod_{i=1}^s p_i$ be squarefree and $\lambda(n) = (n-1)/d$ with d odd. Put*

$$e_i = v_2(p_i - 1), \quad E = \max_i e_i, \quad a = \#\{i : e_i = 1\}.$$

Then $v_2(n-1) = E$. If $E = 1$, then $a = s$ is odd; if $E \geq 2$, then a is even.

Proof. Because d is odd, $v_2(n-1) = v_2(\lambda(n)) = E$. Also $e_i = 1$ exactly when $p_i \equiv 3 \pmod{4}$, so $n \equiv (-1)^a \pmod{4}$. The two cases follow. $\square$

Theorem 9.5 (Exact spectral line at $d = 33$).

$$\mathcal{E}(33) = \{1047619, 2210671, 3494701, 15197491, 25158871, 93755971\}.$$

In particular $N_{\max}(33) = 93\,755\,971$.

Proof. The branch formula gives $B(33) = 166 = 31 + 135$, separating 31 branches with $\kappa = 1$ from 135 with $\kappa > 1$. In the $s = 6$, $\kappa = 1$ branch, $H = 32$ and $\sum e_i - E = 5$, so the five nonmaximal exponents are all 1; Lemma 9.4 gives a contradiction. In the $s = 5$, $\kappa = 1$, $H = 32$ branch, four positive nonmaximal exponents sum to 5, so their multiset is $\{2, 1, 1, 1\}$; here $E \geq 2$, leaving exactly three primes congruent to 3 (mod 4), again contradicting the lemma. The program `certify_d33_reduced.jl` exhausts the remaining 29 $\kappa = 1$ branches and finds none. Independently, `certify_d33_noncar.jl` exhausts all 135 branches with $\kappa > 1$ and returns exactly the six displayed integers. Step (1) of Proposition 9.2 gives no additional prime-power solution at $d = 33$. This exhausts all strata. $\square$

The six surviving points are all squarefree and have exactly three prime factors. Their branch invariants are:

n	factorization	κ	H	j	$\lambda(n)$	$L = (n - 1)/33$
1 047 619	$79 \cdot 89 \cdot 149$	4	8	32	126 984	31 746
2 210 671	$59 \cdot 89 \cdot 421$	4	8	32	267 960	66 990
3 494 701	$7 \cdot 101 \cdot 4943$	7	4	28	741 300	105 900
15 197 491	$37 \cdot 431 \cdot 953$	4	8	32	1 842 120	460 530
25 158 871	$41 \cdot 173 \cdot 3547$	4	8	32	3 049 560	762 390
93 755 971	$41 \cdot 167 \cdot 13693$	4	8	32	11 364 360	2 841 090

Thus this dyadic transition permits six prime factors structurally, but none of the actual exceptions uses more than three; in particular the entire $\kappa = 1$ (Carmichael-index) slice is empty at $d = 33$.

The branch $\kappa = 1$ is exactly the Carmichael-index problem $n - 1 = d\lambda(n)$ studied in the small-index Carmichael literature; compare Pinch [25]. For external comparison, Pinch's exhaustive Carmichael-number tables were extended from the earlier 10^{15} computation to a complete enumeration through 10^{21} [24]. The present exclusion of the $\kappa = 1$ line at $d = 33$ does not rely on absence from Pinch's table: the two largest branches are eliminated analytically by Lemma 9.4, and the remaining branches are exhausted by the certified exact search below. Pinch's table is therefore an independent large-range check rather than an input to the proof.

9.2 Extension from $d = 34$ through $d = 64$

The interval $34 \leq d \leq 64$ has a useful structural split. If $\kappa > 1$, then

$$2^{s-1}\kappa \leq d - 1 \leq 63$$

forces $s \leq 5$. If $\kappa = 1$, then $K = 1$ and

$$n - 1 = d\lambda(n),$$

so the number is precisely a Carmichael number of Carmichael index d . We independently implement the finite prescribed-index recursion for this $\kappa = 1$ stratum. At each prefix the exact ratio bound and the necessary overlap condition restrict the next prime; with two primes left we use Lemma 9.1, factoring the resulting integer T and reconstructing every admissible divisor pair. The two largest empty cases, $d = 34$ and $d = 35$, are recorded as disjoint exact prefix partitions in the supplement; the union of each partition is the full branch, not a size cutoff. All factor outputs are reconstructed and their prime factors are independently certified. The completed computation agrees exactly with Pinch's published prescribed-index list through 64 [25], so Pinch is an external cross-check rather than a proof dependency. The nonempty new $\kappa = 1$ rows occur at

37, 39, 43, 44, 45, 47, 48, 49, 50, 52, 53, 54, 55, 60, 61.

For $\kappa > 1$ we used a second exact implementation of Proposition 9.2, written in C++ with arbitrary-precision integers for the branch algebra. Every exact recursive prime bound encountered in the completed $34 \leq d \leq 64$ runs lies below 2^{64} ; the program aborts rather than truncating a branch if this condition fails. Primality is tested with the first twelve prime Miller–Rabin bases $2, 3, \dots, 37$. Sorenson–Webster proved that the smallest composite passing all twelve is $318665857834031151167461 > 2^{64}$ [35]; hence every C++ primality decision in the recorded 64-bit domain is deterministic. The implementation exhausts all such branches. The only noticeably expensive branch is

$$(d, s, \kappa, H, P) = (49, 5, 3, 16, 1).$$

For it we stop after three primes and apply Lemma 9.1. There are exactly 49 496 surviving three-prime prefixes; exact divisor-pair completion yields no solution. The source and captured log are supplied. Prime powers are enumerated separately by step (1) of Proposition 9.2.

Theorem 9.6 (Certified composite spectrum through $d = 64$). *For every $1 \leq d \leq 64$, the inclusive composite spectral line $\mathcal{E}(d)$ is exactly the corresponding row of Table 2. Across the first 64 lines there are 351 incidences representing 234 distinct odd composites. The largest row size is*

$$\max_{d \leq 64} e(d) = 13,$$

attained at $d = 53$ and $d = 62$.

Proof. The range $d \leq 8$ is Theorem 8.1, the range $9 \leq d \leq 32$ is Theorem 9.3, and $d = 33$ is Theorem 9.5. For $34 \leq d \leq 64$, the prime-power stratum is exhausted by step (1) of Proposition 9.2; the $\kappa = 1$ stratum is exhausted by the independent prescribed-index certificate just described (with Pinch used only as a cross-check); and every $\kappa > 1$ branch is exhausted by the exact search just described, with the single $d = 49$ branch completed by the divisor-pair identity. The accompanying verifier recomputes $\lambda(n)$ and both defining divisibilities for all 351 incidences and returns 234 distinct integers. Hence no stratum is omitted. $\square$

9.3 Reproducibility of the computer-assisted part

The computational claims are supplied with machine-readable supplementary material rather than being treated as an opaque numerical experiment. The reference implementation uses exact integer arithmetic for every pruning decision and performs the final checks

$$d \mid n - 1, \quad (n - 1)/d \mid \lambda(n)$$

for every reported candidate. Failed tasks are required to abort a run.

The original frozen $d \leq 33$ subarchive has SHA-256

bc7b993c4a58678de2aa2303b4e011c8d5a319c24ad805ff00515452f47881b2.

It is retained unchanged inside the extended supplement. The present arXiv supplement additionally contains the exact $34 \leq d \leq 64$ C++ search, the specialized $d = 49$ divisor-pair certificate, the independent $\kappa = 1$ prescribed-index certificate and its exact split logs, the Pinch cross-check, the combined machine-readable $1 \leq d \leq 64$ table, its arithmetic verifier, the exact prime-mass calculation through 64, and the arXiv-only low- ω and OEIS data. A machine-readable SHA manifest identifies every file. The legacy subarchive contains the following proof-critical files:

file	SHA-256
overlap_U_search.jl	df97f12dbf32222991131f36482fece2ab752643638691c36a1fbf37c45ae426
verify_overlap_U_output.jl	670adb590865eca6eb4b9f1154ccdee87e9518a8c2180ec051cfe9ee113bf17
certify_d33_reduced.jl	b023717245e965b16559edc4a0e6d211c34bd82274d3409fa510f1dc93741da7
certify_d33_noncar.jl	57a26bb49947a99a6a712ac2e0c25bf65b033bff3bb0ccb58167647df03d0f08
prime_mass_d33_check.jl	dd3fd20d28a5160615da798fa40d0a7d642ac6ae9191b5ec41ad0b94d8eda212
branch_parameter_counts_d9_d33.tsv	60ead43be1cea3c1af6ab85d8590dd3cf5330864f5b74a8c763585ee8440508d

The frozen reference computation was executed under Julia 1.12.6 on August 23, 2026. The $d \leq 8$ regression self-test passed. The general exact search then exhausted every parameter branch for $9 \leq d \leq 32$ and reproduced the corresponding rows of Table 2; the separate arithmetic verifier checked all 96 resulting incidences. It uses no external packages, deterministically certifies every reported prime factor below 2^{64} , and recomputes n , $\phi(n)$, $\lambda(n)$, L , κ , H , the repeated-prime factor P , and all defining divisibilities from the reported factorizations.

The dyadic transition $d = 33$ is certified by the split proof described above, rather than by an unspecialized rerun of the general search. After the two $\kappa = 1$ branches eliminated analytically by Lemma 9.4 are removed, `certify_d33_reduced.jl` exhausts all 29 remaining $\kappa = 1$ parameter branches and finds none. Independently, `certify_d33_noncar.jl` exhausts all 135 branches with $\kappa > 1$ and returns exactly the six integers in Theorem 9.5; it then checks the factorizations and the $(\kappa, H, j, \lambda, L)$ data displayed above. The reference runs used the command-line option `--resume=false` for a clean non-resuming run. The supplementary branch-count table records the exact parameter-branch counts for every $9 \leq d \leq 33$.

For $34 \leq d \leq 64$, the supplement contains the exact C++ branch search, the independent $\kappa = 1$ prescribed-index certificate, its exact split logs for the two dominant empty indices 34 and 35, the Pinch cross-check, the specialized $d = 49$ last-two-prime certificate, and an independent Python arithmetic verifier. The verifier recomputes $\lambda(n)$ and the defining divisibilities for all 351 incidences and returns 234 distinct integers. Every recursive prime bound encountered in the completed $\kappa > 1$ runs lies below 2^{64} ; this is not an upper bound on n , and the search aborts rather than truncating a branch if a required prime bound exceeds the deterministic primality range. The recorded runs also print the maximum exact recursive prime bound and the maximum primality-test input, together with an explicit `overflow=false` status.

The density calculation used in Theorem 12.5 is certified separately by `prime_mass_d64_check_exact.py`. It checks all $2^{18} - 1 = 262143$ squarefree kernels supported on primes through 61 using exact integer/rational arithmetic, proves the $q \geq 67$ tail exclusion by $G_{64} - \mathfrak{M}_2(64) > G_{64}/4421$, and reproduces both quoted masses and the unique minimizing squarefree kernel 2. Captured output is included in the supplement.

The legacy Julia source uses only standard libraries. The new C++ search tests primality with the first twelve prime bases $2, 3, \dots, 37$ and aborts instead of silently changing to a probable-prime decision outside its 64-bit search domain. Sorenson–Webster’s value $\psi_{12} = 318665857834031151167461$ proves every such C++ decision [35]. The specialized $d = 49$ divisor-pair certificate checks complete factorizations of the integers T factor by factor with the same certificate. The independent $\kappa = 1$ program uses ψ_{12} and, for a few larger auxiliary factors, Sorenson–Webster’s $\psi_{13} = 3317044064679887385961981$; above that it falls back to a recursive Pocklington certificate after an exact factorization of $n - 1$. Thus no probable-prime assumption occurs in either certificate. The separate verifier is an arithmetic verifier of reported outputs; it does not replace the completeness proof in Proposition 9.2. This explicit separation between theorem, search engine, specialized $d = 33$ certification, and independent arithmetic verification is especially important because an earlier exploratory program used integer division before checking $d \mid n - 1$ and consequently produced several false positives. Those exploratory outputs are not used in any theorem of the present paper.

The exact supplement accompanying this revision is frozen and identified by its SHA–256 manifest. The unchanged archive is intended for deposition in Zenodo or an institutional repository; the resulting persistent identifier will replace the placeholder [SUPPLEMENT-DOI] in the archival version. The deposited archive itself will not subsequently be modified.

Table 1: Certificate summary for the new $34 \leq d \leq 64$ computation. “Max bound” is the largest exact recursive prime bound recorded by the component; every recorded overflow/abort flag is false.

component	program / theorem	range	work	max bound	status
$\kappa = 1$ prescribed-index	ratio recursion + Lemma 9.1	$d = 34$	15 branches; 344284 prefixes	2140591	pass
$\kappa = 1$ prescribed-index	same	$d = 35$	15 branches; 381524 prefixes	7296343	pass
$\kappa = 1$ prescribed-index	same	$36 \leq d \leq 48$	227 branches; 202725 prefixes	1113271	Pinch match
$\kappa = 1$ prescribed-index	same	$49 \leq d \leq 64$	384 branches; 194688 prefixes	305758	Pinch match
$\kappa > 1$ C++	Proposition 9.2; 12-base MR	$34 \leq d \leq 48$	2645 branches	4864229	overflow=false
$\kappa > 1$ C++	same	$50 \leq d \leq 54$	1360 branches	22551385	overflow=false
$\kappa > 1$ C++	same	$55 \leq d \leq 59$	1523 branches	1368683	overflow=false
$\kappa > 1$ C++	same	$60 \leq d \leq 64$	1725 branches	31170	overflow=false
$d = 49$ special branch	Lemma 9.1	$(49, 5, 3, 16, 1)$	49496 three-prime prefixes	—	pass

Table 2: Certified inclusive composite spectrum through $d = 64$. Here $e(d) = |\mathcal{E}(d)|$, $g(d)$ counts first (genuine) occurrences, $C(d)$ is the cumulative number of distinct composites, and * marks inheritance from a smaller spectral index.

d	$e(d)$	$g(d)$	$C(d)$	$\mathcal{E}(d)$
1	0	0	0	—
2	0	0	0	—
3	0	0	0	—
4	1	1	1	9
5	1	1	2	6601
6	1	1	3	25
7	2	2	5	15, 561
8	2	1	6	9*, 49
9	3	3	9	30889, 88561, 534061
10	2	1	10	21, 6601*
11	1	1	11	45
12	2	1	12	25*, 121
13	3	3	15	27, 126673, 1333333
14	3	1	16	15*, 169, 561*
15	2	1	17	91, 6601*
16	3	2	19	33, 49*, 65
17	2	2	21	35, 139231
18	7	4	25	289, 30889*, 88561*, 534061*, 55462177, 8885251441, 42018333841
19	4	4	29	39, 153, 4371, 7107
20	4	2	31	21*, 361, 3201, 6601*
21	7	7	38	85, 8695, 10585, 26335, 9717247, 16666651, 37769887
22	3	2	40	45*, 133, 2465
23	5	5	45	231, 645, 1105, 24013, 34133
24	4	1	46	25*, 49*, 121*, 529
25	4	3	49	51, 6601*, 15051, 11921001
26	4	1	50	27*, 105, 126673*, 1333333*
27	6	3	53	55, 325, 3565, 30889*, 88561*, 534061*
28	3	1	54	57, 169*, 561*
29	6	6	60	117, 175, 1045, 1447681, 2121061, 164424781
30	4	2	62	91*, 841, 6601*, 158401
31	8	8	70	63, 125, 435, 62745, 1154131, 2609581, 3936691, 5624641

d	$e(d)$	$g(d)$	$C(d)$	$\mathcal{E}(d)$
32	6	4	74	33*, 65*, 385, 961, 12673, 12801
33	6	6	80	1047619, 2210671, 3494701, 15197491, 25158871, 93755971
34	4	2	82	35*, 69, 341, 139231*
35	4	3	85	561*, 91001, 610051, 683761
36	9	2	87	145, 217, 289*, 30889*, 88561*, 534061*, 55462177*, 8885251441*, 42018333841*
37	11	11	98	75, 17169, 33153, 45141, 62605, 2415953, 2929291, 11972017, 16894571, 67902031, 191834567
38	9	5	103	39*, 77, 153*, 1065, 1369, 4371*, 7107*, 13833, 1170401
39	5	3	106	703, 15211, 126673*, 334153, 1333333*
40	7	4	110	81, 361*, 481, 3201*, 6601*, 460801, 1372801
41	8	8	118	165, 247, 2871, 8365, 73555, 184255, 9345541, 128950249
42	9	1	119	85*, 169*, 1681, 8695*, 10585*, 26335*, 9717247*, 16666651*, 37769887*
43	7	7	126	87, 259, 861, 22791, 52633, 69231, 82733
44	5	2	128	45*, 133*, 1849, 2465*, 15841
45	7	4	132	91*, 451, 8911, 88561*, 534061*, 3763801, 14245741
46	8	3	135	93, 185, 231*, 369, 645*, 1105*, 24013*, 34133*
47	9	9	144	95, 1035, 2821, 4795, 204263, 223721, 308039, 559489, 38649041
48	5	3	147	49*, 529*, 1729, 2209, 52417
49	12	12	159	99, 87676681, 133861729, 161005573, 258844951, 1411040359, 3462426241, 7950140448251, 131097700645913, 798178197617711, 989473937059427, 1208361237478669
50	9	5	164	51*, 301, 1001, 6601*, 15051*, 3679201, 11921001*, 4199932801, 1353385637801
51	3	2	166	205, 32131, 139231*
52	8	5	171	105*, 833, 9361, 126673*, 741937, 1333333*, 206955841, 560630848961
53	13	13	184	425, 637, 3605, 8481, 55651, 3547291, 3984541, 43523707, 52519291, 260410837, 1271325841, 152674725071, 3916268927681
54	8	2	186	55*, 325*, 2809, 3565*, 30889*, 88561*, 534061*, 4169867689
55	10	9	195	111, 221, 1431, 6601*, 15181, 100101, 689701, 1269621, 271794601, 568898551
56	6	3	198	57*, 169*, 225, 561*, 25761, 376993
57	6	6	204	115, 343, 2737, 95761, 37883341, 713798461
58	9	3	207	117*, 175*, 1045*, 2553, 80041, 1447681*, 2121061*, 8372881, 164424781*
59	4	4	211	119, 1653, 1771, 42353741
60	8	3	214	121*, 361*, 841*, 3481, 6601*, 25201, 158401*, 6840001
61	12	12	226	123, 245, 1221, 44409, 50997, 74665, 631351, 964411, 1102759, 940724555, 1962804565, 2417562005

d	$e(d)$	$g(d)$	$C(d)$	$\mathcal{E}(d)$
62	13	5	231	$63^*, 125^*, 435^*, 3721, 19345, 62745^*, 264121, 828073, 1154131^*, 2609581^*, 3936691^*, 5624641^*, 7697921$
63	7	1	232	$1891, 8695^*, 10585^*, 26335^*, 9717247^*, 16666651^*, 37769887^*$
64	7	2	234	$65^*, 129, 385^*, 961^*, 1281, 12673^*, 12801^*$

Theorem 9.7 (Exact primality criterion through spectral index 64). *Let $n > 1$ be odd, let $\gcd(b, n) = 1$, and suppose the exact order $L = \text{ord}_n(b)$ satisfies*

$$L = \frac{n-1}{d}, \quad 1 \leq d \leq 64.$$

Then either n is prime or n belongs to the row $\mathcal{E}(d)$ of Table 2. In particular, if the row information is ignored, there are 234 distinct composite possibilities in the whole certified range.

Proof. For $d \leq 8$ this is the small-index classification of Section 8; for $9 \leq d \leq 32$ it is Theorem 9.3; $d = 33$ is Theorem 9.5; and $34 \leq d \leq 64$ is covered by Theorem 9.6. $\square$

Remark 9.8 (How to use the criterion). *This statement is stronger than a probabilistic test. Once L is known exactly and $d \leq 64$, primality is decided by a finite table lookup: if a composite candidate is not in the relevant row, it is prime. The theorem does not claim that computing L from scratch is cheap.*

10 Quantitative rarity and prime–pseudoprime spectral separation

Fixed-index finiteness is only the starting point. To measure how sparse the low-index composite spectrum actually is, define

$$C(D) = \left| \bigcup_{1 \leq d \leq D} \mathcal{E}(d) \right|, \quad C^*(D) = \left| \bigcup_{1 \leq d \leq D} \mathcal{E}^*(d) \right|.$$

The first quantity uses the inclusive spectral convention; the second uses Somer’s nontrivial-base convention.

Theorem 10.1 (Quantitative rarity through $d = 64$). *The certified table gives*

$$C(64) = 234, \quad C^*(64) = 231.$$

Counting inherited reappearances separately, the first 64 inclusive lines contain 351 spectral incidences, while Somer’s nontrivial-base convention contains 329.

The difference between the inclusive and nontrivial-base counts can again be audited directly.

The 22 excluded incidences are

d	excluded occurrence
4	9 ($L = 2$)
8	9 ($L = 1$)
12	25 ($L = 2$)
13	27 ($L = 2$)
14	15 ($L = 1$)
20	21 ($L = 1$)
24	25 ($L = 1$), 49 ($L = 2$)
26	27 ($L = 1$)
32	33 ($L = 1$)
34	35 ($L = 1$)
38	39 ($L = 1$)
40	81 ($L = 2$)
44	45 ($L = 1$)
48	49 ($L = 1$)
50	51 ($L = 1$)
54	55 ($L = 1$)
56	57 ($L = 1$)
60	121 ($L = 2$)
62	63 ($L = 1$), 125 ($L = 2$)
64	65 ($L = 1$).

Order 1 is represented only by $1 \bmod n$, and for an odd prime power the unique involution is $-1 \bmod n$. Thus these are precisely the incidences removed by Somer's convention. Only 9, 27, 81 disappear completely from the union through 64; all other affected integers have another nontrivial spectral occurrence by that cutoff. Hence $351 - 22 = 329$ incidences and $234 - 3 = 231$ distinct composites.

The distinction between 234 and 351 is important: a composite can occupy several spectral lines by the inheritance theorem. The title of this paper refers to the certified scarcity of the first quantity across a low-index range; the general upper bounds proved later are much weaker than the observed values. "Rare" is therefore not being used as a synonym for Somer's theorem that each single line is finite; it refers to the observed and certified scarcity of distinct composites across an entire low-index interval.

Figure 1 shows the cumulative composite count. The growth is irregular, with visible changes near the dyadic thresholds at which one more distinct prime factor becomes structurally possible.

On the prime side the relevant comparison is not the number of primes on one line, since every existential prime line is infinite, but the *fixed-base prime mass* carried by the same low-index interval. This is quantified in Theorem 12.5 below. The resulting juxtaposition is a structural spectral comparison: the existential-over-bases composite union remains thin in the certified range, while each fixed non-perfect-power base places most of its prime residual-index mass in the same range.

The two sides of this display are deliberately not probabilities on the same sample space. The composite set $\mathcal{E}(d)$ is enlarged by allowing the base to vary with n , whereas $\delta(b, d)$ fixes one base. The comparison is therefore structural rather than a direct density identity: even the union over all possible bases contains very few low-index composite exceptions, while each individual positive non-perfect-power base places most of its prime residual-index mass in the same spectral range. The line-sampling probability introduced in Section 11 later places primes and composites in one common sample space.

The computations also suggest a dyadic stratification. Composite spectral lines become combinatorially richer when d passes

$$d = 2^r + 1,$$

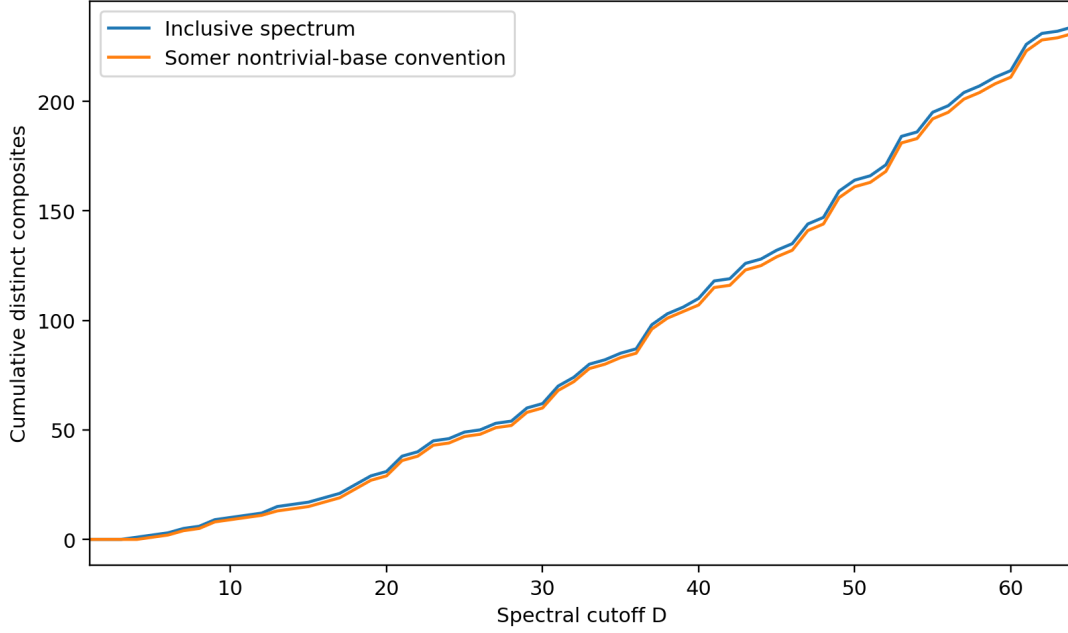


Figure 1: Cumulative number of distinct odd composite exceptions through spectral index $D \leq 64$. The upper curve uses the inclusive group-theoretic spectrum $\mathcal{E}(d)$; the lower curve uses Somer’s nontrivial-base convention $\mathcal{E}^*(d)$.

because another distinct prime factor becomes compatible with

$$2^{s-1} \leq d - 1.$$

11 Counting composites on a spectral line and a primality probability

The exact low-index table suggests a natural quantitative question stronger than fixed- d finiteness. Put

$$e(d) := |\mathcal{E}(d)|, \quad g(d) := |\mathcal{G}(d)|,$$

and, for the stratification by the number of distinct prime factors,

$$e_s(d) := |\{n \in \mathcal{E}(d) : \omega(n) = s\}|.$$

No convincing asymptotic formula for $e(d)$ or $g(d)$ is presently known. The data through 64 are much too short and irregular to justify one. Nevertheless the structural theory gives considerably more than the crude consequence of fixed-index finiteness.

11.1 The crude bound inherited from the size theorem

For $d \geq 2$, every composite in $\mathcal{E}(d)$ satisfies

$$n < U_0(d) := (dS(d))^{2M(d)}$$

and $n \equiv 1 \pmod{d}$. Hence

$$e(d) \ll \frac{U_0(d)}{d}, \quad \log e(d) \leq 2d(\log d + \log \log d + O(1)).$$

This is uniform but enormously larger than the observed counts. We now obtain a sharper bound by counting factorization patterns directly rather than counting all integers below the size obstruction.

11.2 Overlap-resolved branches versus master-equation branches

The exact search is organized by (s, κ, H, P) . Its number of *overlap-resolved* parameter branches is

$$B(d) = \sum_{s=2}^{S(d)} \sum_{\kappa \leq (d-1)/2^{s-1}} \left\lfloor \frac{d-1}{2^{s-1}\kappa} \right\rfloor \tau(\kappa).$$

This decomposition is computationally valuable because fixing κ and H permits exact λ -overlap pruning.

Proposition 11.1 (Asymptotic overlap-resolved branch count). *As $d \rightarrow \infty$,*

$$B(d) = \frac{1}{2}d(\log d)^2 + O(d \log d).$$

Proof. Put $D = d - 1$ and $r = s - 1$. For $X_r = \lfloor D/2^r \rfloor$, the inner sum is

$$\sum_{\kappa \leq X_r} \tau(\kappa) \left\lfloor \frac{X_r}{\kappa} \right\rfloor = \sum_{m \leq X_r} d_3(m),$$

where $d_3 = 1 * 1 * 1$. The Piltz divisor estimate

$$\sum_{m \leq x} d_3(m) = \frac{1}{2}x(\log x)^2 + O(x \log x)$$

is more than sufficient here; see, for example, [15, Chapter 1]. Summing over $r \geq 1$ and using $\sum 2^{-r} = 1$ together with convergence of $\sum r2^{-r}$ and $\sum r^2 2^{-r}$ gives the result. $\square$

For the Diophantine equation itself, however, κ and H occur only through

$$j = \kappa H, \quad c = \frac{j}{P}.$$

This collapses one logarithmic layer of the parameter space.

Proposition 11.2 (Relaxed collapsed master-equation count). *Fix $s \geq 2$ and put $D = d - 1$. Every actual master-equation parameter pair (j, P) satisfies*

$$1 \leq j \leq d - 1, \quad 2^{s-1}P \mid j.$$

If we deliberately drop the additional arithmetic restrictions on P —in particular that $P = n/\text{rad}(n)$ is odd and is supported on the primes dividing n —the number of resulting relaxed candidate pairs is

$$B_{\text{mast},s}^+(d) = \sum_{m \leq D/2^{s-1}} \tau(m).$$

Consequently the number of actual (j, P) pairs is at most $B_{\text{mast},s}^+(d)$, and the total relaxed count is

$$B_{\text{mast}}^+(d) = \sum_{s=2}^{S(d)} B_{\text{mast},s}^+(d) = \sum_{r \geq 1} \sum_{m \leq D/2^r} \tau(m).$$

Moreover

$$B_{\text{mast}}^+(d) = D \log D + (2\gamma - 1 - 2 \log 2)D + O(\sqrt{D}) = d \log d + O(d), \quad (19)$$

where γ is Euler's constant.

Proof. Every actual branch has $P \mid \kappa$ and $2^{s-1} \mid H$, hence $2^{s-1}P \mid j = \kappa H$. Writing $j = 2^{s-1}m$, every actual P is therefore a divisor of m . If all divisors of m are retained, without checking oddness, prime support, or whether a compatible λ -overlap really exists, there are exactly $\tau(m)$ relaxed choices. This proves the upper count.

The classical divisor summatory formula

$$\sum_{m \leq x} \tau(m) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$$

now gives, after summing over $r = s - 1 \geq 1$,

$$D \log D \sum_{r \geq 1} 2^{-r} - D \log 2 \sum_{r \geq 1} r 2^{-r} + (2\gamma - 1)D \sum_{r \geq 1} 2^{-r} + O\left(\sqrt{D} \sum_{r \geq 1} 2^{-r/2}\right).$$

Since the first two geometric sums are 1 and 2, respectively, (19) follows. $\square$

Thus the extra factor of order $\log d$ in the overlap-resolved search count $B(d)$ disappears already after forgetting the particular decomposition $j = \kappa H$. The relaxed quantity $B_{\text{mast}}^+(d)$ is not claimed to count realizable arithmetic branches exactly; it is a convenient and safe upper layer for estimating how many Diophantine candidates a spectral line can support.

11.3 Exact low- ω strata

The first three strata admit particularly transparent estimates.

Proposition 11.3 (Prime-power stratum). *For an odd prime power $n = p^a$, $a \geq 2$, put*

$$R_a(p) = 1 + p + \cdots + p^{a-1} = \frac{p^a - 1}{p - 1}.$$

Then

$$d_\lambda(p^a) = R_a(p), \quad \text{Spec}_L(p^a) = \{R_a(p)r : r \mid p - 1\}.$$

Consequently

$$e_1(d) = \#\{(p, a) : p \text{ odd prime}, a \geq 2, R_a(p) \mid d, d/R_a(p) \mid p - 1\},$$

and

$$e_1(d) = O(\tau(d) \log d) = d^{o(1)}.$$

Proof. Since

$$\lambda(p^a) = p^{a-1}(p - 1), \quad p^a - 1 = (p - 1)R_a(p),$$

and $R_a(p) \equiv 1 \pmod{p}$, one has $\gcd(p^a - 1, \lambda(p^a)) = p - 1$. The formula for the genuine index follows, and spectral inheritance gives the displayed spectrum. Finally $R_a(p) \geq (3^a - 1)/2$, so only $O(\log d)$ exponents can occur; for each exponent the strictly increasing function $R_a(p)$ can take at most one prime value for each divisor of d . $\square$

For two distinct prime factors, Lemma 9.1 has no prefix: $A = B = 1$, $a = d$, $b = j$, and $\Delta = d - j$. It therefore becomes

$$((d - j)p - d)((d - j)q - d) = c[d(P - 1) + j]. \quad (20)$$

Indeed, since $cP = j \leq d - 1$ and $c, j \leq d - 1$,

$$T = c[d(P - 1) + j] = dj - cd + cj = O(d^2).$$

Theorem 11.4 (One-, two-, and three-prime counting bounds). *As $d \rightarrow \infty$,*

$$\boxed{e_1(d) \leq d^{o(1)}, \quad e_2(d) \leq d^{1+o(1)}, \quad e_3(d) \leq d^{1+o(1)}}.$$

Proof. The first assertion is Proposition 11.3. For $s = 2$, Proposition 11.2 gives $O(d \log d)$ relaxed master parameter pairs, and hence at most that many actual pairs. Equation (20) has $T = O(d^2)$ as noted above, so it supplies at most $\tau(T)/2 = d^{o(1)}$ divisor-pair completions on each branch, by the standard maximal-order estimate

$$\tau(m) = \exp\left(O\left(\frac{\log m}{\log \log m}\right)\right).$$

This proves $e_2(d) \leq d^{1+o(1)}$.

For $s = 3$, $4 \mid H$ and $P \mid \kappa$, hence

$$P \mid \kappa \mid \frac{\kappa H}{4} = \frac{j}{4}.$$

Write $j = 4m$; then $P \mid m$. If $p = p_1$, the smallest-prime bound gives

$$p < \frac{d + 2j}{d - j}, \quad \text{hence} \quad j > \frac{d(p - 1)}{p + 2}.$$

Also $p < 3d$. For fixed p , the number of multiples of 4 in the allowed j -interval is $O(d/p + 1)$, and for each j there are at most $\tau(j/4) = d^{o(1)}$ choices for P . Mertens' estimate for reciprocal primes therefore gives at most

$$d^{o(1)} \sum_{p < 3d} \left(\frac{d}{p} + 1\right) = d^{1+o(1)}$$

possible triples (j, P, p_1) . For each such prefix $p_1 < 3d$, so $A = p_1 - 1 = O(d)$ and $B = p_1 = O(d)$. Hence in Lemma 9.1

$$a = O(d^2), \quad b = O(d^2), \quad c = O(d), \quad T = O(d^4),$$

and there are only $d^{o(1)}$ divisor-pair completions. This proves the third assertion. $\square$

These estimates are intentionally one-sided. They do not suggest that $e_2(d)$ or $e_3(d)$ actually has polynomial order; the certified small-index data are far smaller. Their point is that the first nontrivial strata already admit nearly linear rigorous upper bounds.

11.4 An exploratory exact count of the low- ω strata

To see the scale hidden by the upper bounds, we implemented the preceding prime-power formula and divisor-pair identities independently in Python/SymPy. Every generated candidate is rechecked from n , d , and $\lambda(n)$ rather than accepted from the parametrization alone. As a regression test, the program reproduces every $\omega(n) \leq 3$ incidence in the certified spectrum through $d = 64$. It was then run for every $2 \leq d \leq 500$.

range of d	mean $e_1(d)$	mean $e_2(d)$	mean $e_3(d)$	mean $\sum_{s \leq 3} e_s(d)$
2–100	0.556	1.424	4.475	6.455
101–200	0.510	2.020	9.710	12.240
201–500	0.420	2.350	13.817	16.587

These are *not* values of the full function $e(d)$ beyond the certified range: strata with four or more distinct prime divisors are omitted. Through $d = 64$, however, the low- ω calculation accounts for 320 of the 351 certified spectral incidences: 40 have one distinct prime factor, 71 have two, and 209 have three. The remaining 31 incidences have $\omega(n) = 4$; none in the certified range has five or more distinct prime factors. The experiment therefore supports the qualitative impression that the actual low-order strata grow far more slowly than the worst-case bounds. We deliberately do not fit an asymptotic law to the range $d \leq 500$: it is too short, and the dyadic changes in the allowed value of $\omega(n)$ make extrapolation particularly unsafe. The script and captured output are supplied separately as exploratory material; they are not used in any proof.

11.5 A uniform upper bound for the full composite line

The same two-prime completion identity improves the global count even when s is allowed to grow with d .

Theorem 11.5 (Uniform counting bound for a spectral line). *As $d \rightarrow \infty$,*

$$\boxed{\log e(d) \leq \frac{d}{2} (\log d + \log \log d + O(1))}. \quad (21)$$

Equivalently,

$$e(d) \leq \exp \left\{ \frac{d}{2} (\log d + \log \log d + O(1)) \right\}.$$

Proof. The strata $s = 1, 2$ are already smaller than the claimed bound, so suppose $s \geq 3$ and fix one overlap-resolved branch (s, κ, H, P) . Let

$$Q_r := \prod_{i=1}^r p_i$$

be the radical of a prime prefix. The smallest-prime estimate gives $Q_1 = p_1 < sd$. For $k \geq 2$, if $p_k > \kappa$, the squarefree-suffix bound applies. Since $K_{k-1} \mid \kappa$,

$$\lambda(P_{k-1}) \leq \phi(P_{k-1}) = K_{k-1} \prod_{i < k} (p_i - 1) < K_{k-1} Q_{k-1} \leq \kappa Q_{k-1}.$$

Therefore

$$p_k < d\lambda(P_{k-1})(s - k + 1) \leq d\kappa s Q_{k-1}.$$

If $p_k \leq \kappa$, the same weaker inequality is automatic. Hence

$$Q_k < d\kappa s Q_{k-1}^2.$$

Induction yields

$$Q_r < (sd)^{2^r - 1} \kappa^{2^{r-1} - 1}. \quad (22)$$

Stop with $r = s - 2$ primes selected. Since $2^{s-1} \kappa \leq d - 1$, one has

$$2^r = 2^{s-2} \leq \frac{d-1}{2\kappa} < \frac{d}{2\kappa}, \quad 2^{r-1} = 2^{s-3} < \frac{d}{4\kappa}.$$

Consequently the two leading logarithmic contributions in (22) satisfy

$$(2^r - 1) \log(sd) \leq \frac{d}{2\kappa} (\log d + \log s), \quad (2^{r-1} - 1) \log \kappa \leq \frac{d}{4} \frac{\log \kappa}{\kappa}.$$

Now $s = O(\log d)$ and $(\log \kappa)/\kappa$ is bounded for $\kappa \geq 1$; the first term is largest in leading order at $\kappa = 1$. Thus (22) gives

$$\log Q_r \leq \frac{d}{2}(\log d + \log \log d + O(1)). \quad (23)$$

On the fixed branch, the numerical value of P is already fixed. Hence an increasing prime prefix is determined by the squarefree integer Q_r ; for any prefix prime dividing P , its exponent is fixed by $v_p(P) + 1$. The number of possible prefixes on this branch is therefore at most the right-hand size bound for Q_r itself. Required prime factors of P that are not yet in the prefix must occur among the final two primes and are handled by the support check in the completion step.

For a fixed prefix, Lemma 9.1 gives at most $\tau(T)/2$ completions. Here

$$a = dA \leq dQ_r, \quad b = jB \leq dQ_r, \quad c \leq d,$$

so

$$T = a(b - c) + bc \leq 2d^2Q_r^2.$$

Thus $\log T = O(d \log d)$ and the maximal-order bound for the divisor function gives $\log \tau(T) = O(d)$, lower order relative to (23). Finally the total number of overlap-resolved branches is only $B(d) = O(d(\log d)^2)$, whose logarithm is also lower order. Summing over all branches proves (21). $\square$

Compared with the crude estimate inherited from the size theorem, the leading coefficient in the exponent has fallen from 2 to $1/2$. This is not merely a cosmetic improvement: after inversion it enlarges the range in which a spectral line is overwhelmingly prime under the sampling model below.

11.6 A natural conditional probability of primality

There is no intrinsic probability that an integer is prime until a sampling model is specified. We choose uniformly from the integers on a fixed existential spectral line d up to a cutoff X :

$$\mathcal{S}_d(X) = \{n \leq X : \exists b, \gcd(b, n) = 1, \text{ord}_n(b) = (n - 1)/d\}.$$

Its prime part is $\{p \leq X : p \equiv 1 \pmod{d}\}$, while its composite part is $\mathcal{E}(d) \cap [1, X]$. Put

$$\pi(X; d, 1) = \#\{p \leq X : p \equiv 1 \pmod{d}\}.$$

Then

$$\Pr_X(n \text{ prime} \mid d) = \frac{\pi(X; d, 1)}{\pi(X; d, 1) + |\mathcal{E}(d) \cap [1, X]|}. \quad (24)$$

Theorem 11.6 (Primality probability on a fixed existential line). *For every fixed $d \geq 1$,*

$$\Pr_X(n \text{ prime} \mid d) = 1 - e(d)\phi(d)\frac{\log X}{X} + o\left(\frac{\log X}{X}\right).$$

In particular

$$\boxed{\Pr_X(n \text{ prime} \mid d) \longrightarrow 1.}$$

Proof. For fixed d and all sufficiently large X , $|\mathcal{E}(d) \cap [1, X]| = e(d)$. The prime number theorem in arithmetic progressions gives

$$\pi(X; d, 1) \sim \frac{\text{Li}(X)}{\phi(d)} \sim \frac{X}{\phi(d) \log X}.$$

Substitution in (24) and expansion of $1/(1 + y)$ prove the formula. $\square$

For fixed d , the convergence to 1 is essentially the prime-number-theorem consequence of Somer's finiteness theorem; the first-order coefficient records how the exact finite composite count enters. The stronger analytic use of the new counting theorem is the growing- d result below.

For $d \leq 64$ the leading coefficient

$$\rho(d) := e(d)\phi(d)$$

is known exactly. Its maximum is

$$\max_{1 \leq d \leq 64} \rho(d) = 720, \quad \text{at } d = 61,$$

since $e(61) = 12$ and $\phi(61) = 60$.

Corollary 11.7 (Uniform line-sampling estimate through 64). *For the finite family $1 \leq d \leq 64$,*

$$\min_{1 \leq d \leq 64} \frac{\Pr(n \text{ prime} \mid d)}{X} = 1 - 720 \frac{\log X}{X} + o\left(\frac{\log X}{X}\right).$$

The lines $d = 1, 2, 3$ have $e(d) = 0$ and therefore contain no odd composites.

The new counting theorem also permits d to grow with the sampling cutoff.

Theorem 11.8 (A uniform high-probability spectral range). *For every fixed $0 < \varepsilon < 2$,*

$$\boxed{\inf_{1 \leq d \leq (2-\varepsilon) \log X / \log \log X} \frac{\Pr(n \text{ prime} \mid d)}{X} \longrightarrow 1 \quad (X \rightarrow \infty).} \quad (25)$$

Proof. Put

$$D_X = (2 - \varepsilon) \frac{\log X}{\log \log X}.$$

Since $D_X \ll \log X$, Siegel–Walfisz gives, uniformly for $d \leq D_X$,

$$\pi(X; d, 1) = \frac{X}{\phi(d) \log X} (1 + o(1));$$

see, for example, [15, Chapter 5]. Choose a constant C and d_0 so that Theorem 11.5 gives

$$\log e(d) \leq \frac{d}{2} (\log d + \log \log d + C) \quad (d \geq d_0).$$

The right-hand side is increasing for sufficiently large d . Hence, uniformly for $d_0 \leq d \leq D_X$,

$$\begin{aligned} \log e(d) &\leq \frac{D_X}{2} (\log D_X + \log \log D_X + C) \\ &= \left(1 - \frac{\varepsilon}{2} + o(1)\right) \log X. \end{aligned}$$

The finitely many $d < d_0$ contribute only bounded constants $e(d)$ and satisfy the same weaker estimate for large X . Therefore

$$e(d) \leq X^{1-\varepsilon/2+o(1)}$$

uniformly throughout the displayed range. Since $\phi(d) \leq d$,

$$\frac{|\mathcal{E}(d) \cap [1, X]|}{\pi(X; d, 1)} \leq X^{-\varepsilon/2+o(1)} d \log X \longrightarrow 0$$

uniformly. Equation (24) proves the claim. □

Remark 11.9 (Certainty versus high probability). *The constants $1/2$ and 2 describe different statements. The asymptotic size theorem gives the deterministic implication*

$$d \leq \left(\frac{1}{2} - \varepsilon\right) \frac{\log n}{\log \log n} \implies n \text{ is prime}$$

for every sufficiently large candidate. Theorem 11.8 extends to the four-times-wider leading range

$$d \leq (2 - \varepsilon) \frac{\log X}{\log \log X},$$

but only in the explicitly specified uniform line-sampling sense: composites are asymptotically negligible rather than impossible. Neither constant is claimed sharp.

For $d \leq 64$, Theorem 9.7 is stronger still: once n and its exact period are known, the explicit exceptional table decides the question rather than assigning a probability.

For a fixed base b , put $e_b(d) = |\mathcal{E}_b(d)|$. Since $\mathcal{E}_b(d) \subseteq \mathcal{E}(d)$, this number is finite. Under the GRH hypotheses of Theorem 12.1, if $\delta(b, d) > 0$, the same fixed- d argument gives

$$\Pr_X(n \text{ prime} \mid b, d) = 1 - \frac{e_b(d)}{\delta(b, d)} \frac{\log X}{X} + o\left(\frac{\log X}{X}\right).$$

This formula makes explicit how the composite count and the fixed-base prime density combine in a spectral estimate of primality.

12 Fixed-base prime spectra and near-primitive roots

The existential prime spectrum of Remark 2.5 allows the base to vary with the prime. A finer spectral problem fixes an integer $b \notin \{-1, 0, 1\}$ and considers

$$\mathcal{P}_b(d) = \left\{ p \nmid b : \text{ord}_p(b) = \frac{p-1}{d} \right\}.$$

In the terminology of the Artin literature, b is then an d -near-primitive root modulo p , and d is the residual index of $b \bmod p$.

The relevant density theory is classical. Following Hooley's method for Artin's conjecture, Wagstaff proved the near-primitive-root density statement under GRH; Moree subsequently derived explicit Euler-product and entanglement correction formulas [37, 21, 22].

Theorem 12.1 (Wagstaff density theorem, with Moree correction formulas, under GRH). *Let $b \in \mathbb{Q} \setminus \{-1, 0, 1\}$ and $d \geq 1$. Assume the Riemann Hypothesis for the Dedekind zeta functions of every Kummer field $\mathbb{Q}(\zeta_{kd}, b^{1/(kd)})$, $k \geq 1$. Then the set $\mathcal{P}_b(d)$ has a natural density*

$$\delta(b, d) = \sum_{k=1}^{\infty} \frac{\mu(k)}{[\mathbb{Q}(\zeta_{kd}, b^{1/(kd)}) : \mathbb{Q}]}. \quad (\text{A1})$$

This density is a rational multiple of Artin's constant. Its vanishing is governed by explicit entanglement conditions.

For a fixed base b and a cutoff D , define the cumulative prime spectral mass

$$\mathfrak{M}_b(D) = \sum_{1 \leq d \leq D} \delta(b, d).$$

Since each prime $p \nmid b$ has one residual index, $\mathfrak{M}_b(D)$ is the natural density, relative to all primes, of those primes for which

$$\frac{p-1}{\text{ord}_p(b)} \leq D.$$

For the spectral comparison it is useful that a large class of bases has an especially simple odd-index spectrum. Write

$$b = b_0^h,$$

where $b_0 > 0$ is not a perfect power in $\mathbb{Q}$ and $h \geq 1$. For a positive integer b that is not a perfect power, define its squarefree part by

$$\text{sf}(b) := \prod_{v_p(b) \text{ odd}} p.$$

This is the squarefree representative of b modulo rational squares and should not be confused with the radical $\text{rad}(b)$.

Definition 12.2 (Odd-index generic base). *A positive integer $b > 1$ is called odd-index generic if it is not a perfect power and*

$$\text{sf}(b) \not\equiv 1 \pmod{4}.$$

Equivalently, the quadratic discriminant of $\mathbb{Q}(\sqrt{b})$ is even.

Lemma 12.3 (Explicit positive non-perfect-power correction factor, under GRH). *Let $b > 1$ be a positive integer that is not a perfect power, put $g = \text{sf}(b)$, and write*

$$d_2 = 2^{v_2(d)}, \quad B(g, d) = \prod_{\substack{q|g \\ q \nmid d}} \frac{-1}{q^2 - q - 1}.$$

Then Moree's $h = 1$ formula depends on b only through the quadratic field $\mathbb{Q}(\sqrt{g})$, and the ratio $\delta(b, d)/E(d)$ is given by

$g \pmod{4}$	d_2	$\delta(b, d)/E(d)$
1	1	$1 - B(g, d)$
1	$d_2 \geq 2$	$1 + B(g, d)$
2	$d_2 < 4$	1
2	$d_2 = 4$	$1 - B(g, d)/3$
2	$d_2 > 4$	$1 + B(g, d)$
3	$d_2 = 1$	1
3	$d_2 = 2$	$1 - B(g, d)/3$
3	$d_2 \geq 4$	$1 + B(g, d)$.

This is the positive $h = 1$ specialization of Moree's explicit near-primitive-root density formula; equivalently it is Table 1 of [22] applied to the squarefree representative of b modulo rational squares.

Derivation of Lemma 12.3. For b not a perfect power, Moree's degree calculation gives

$$[\mathbb{Q}(\zeta_m, b^{1/m}) : \mathbb{Q}] = \frac{m\phi(m)}{\epsilon(m)},$$

where $\epsilon(m) = 2$ precisely when $2 \mid m$ and the quadratic discriminant $D = \text{disc } \mathbb{Q}(\sqrt{\text{sf}(b)})$ divides m ; otherwise $\epsilon(m) = 1$. Insert this dichotomy into (A1). Relative to the generic Euler product $E(d)$, the terms for which the discriminant condition is active force a finite set S of primes into the squarefree Möbius variable. The resulting Euler-product ratio is

$$\prod_{p \in S} \frac{-1}{pc_p - 1}, \quad c_p = \begin{cases} p, & p \mid d, \\ p - 1, & p \nmid d. \end{cases}$$

The 2-adic condition determines the eight cases in the table. In particular when $p = 2 \mid d$ the factor is $-1/(2^2 - 1) = -1/3$. This gives exactly the listed correction factors. $\square$

Corollary 12.4 (Universal odd-index prime spectrum, under GRH). *Let b be odd-index generic and let d be odd. Then*

$$\delta(b, d) = E(d) = \frac{A}{d^2} \prod_{q|d} \frac{q^2 - 1}{q^2 - q - 1}. \quad (\text{A2})$$

where A is Artin's constant. In particular, for odd spectral indices the prime density is independent of the choice of odd-index generic base.

Proof. If d is odd, then $d_2 = 1$. For an odd-index generic base one has $g = \text{sf}(b) \equiv 2$ or $3 \pmod{4}$. The corresponding $d_2 = 1$ rows of Lemma 12.3 both give $\delta(b, d)/E(d) = 1$. $\square$

Theorem 12.5 (Prime-mass minimum and all-base infimum through $d = 64$, under GRH). *Let*

$$G_{64} := \sum_{1 \leq d \leq 64} E(d), \quad \mathfrak{M}_b(64) := \sum_{1 \leq d \leq 64} \delta(b, d).$$

Then:

(i) *The correction-free Wagstaff mass is*

$$G_{64} = 0.984529145953884713 \dots$$

(ii) *Among all odd-index generic bases,*

$$\min_b \mathfrak{M}_b(64) = 0.982709160681865379 \dots$$

The minimum is attained exactly by those bases in this class whose squarefree part is 2.

(iii) *More strongly, the same sharp minimum holds over all positive integer bases $b > 1$ that are not perfect powers:*

$$\mathfrak{M}_b(64) \geq \mathfrak{M}_2(64) = 0.982709160681865379 \dots$$

Equality holds precisely when the squarefree part of b is 2.

(iv) *If arbitrary perfect-power bases are admitted, no positive minimum exists. Instead*

$$\inf_{b \geq 2} \mathfrak{M}_b(64) = 0.$$

A sequence of powers of 2 approaches this infimum.

Proof. Part (i) is the direct sum of Moree's generic factors

$$E(d) = \frac{A}{d^2} \prod_{q|d} \frac{q^2 - 1}{q^2 - q - 1}.$$

For (iii), let $g = \text{sf}(b)$ for a positive non-perfect-power base. Lemma 12.3 reduces the problem to squarefree kernels $g > 1$. Every correction has the form

$$\delta(b, d) = E(d) \{1 + \alpha(g, d) B(g, d)\}, \quad \alpha(g, d) \in \{0, \pm \frac{1}{3}, \pm 1\}.$$

If g contains a prime $q \geq 67$, then $q \nmid d$ for all $d \leq 64$ and

$$|B(g, d)| \leq \frac{1}{67^2 - 67 - 1} = \frac{1}{4421}.$$

Hence

$$|\mathfrak{M}_b(64) - G_{64}| \leq \frac{G_{64}}{4421}.$$

Exact rational arithmetic gives

$$G_{64} - \mathfrak{M}_2(64) > \frac{G_{64}}{4421},$$

so no minimizing kernel contains a prime at least 67.

It remains to inspect the $2^{18} - 1 = 262143$ nonempty squarefree products of

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61.$$

The exact computation evaluates Lemma 12.3 for all of these kernels and finds the unique minimizer $g = 2$. The calculation is implemented without floating-point comparisons in `prime_mass_d64_check_exact.py`; its captured output is included in the supplement. This proves (iii), and (ii) follows because the odd-index generic class contains the kernel 2.

For (iv), put

$$H_y = \prod_{\substack{67 \leq q \leq y \\ q \text{ prime}}} q, \quad b_y = 2^{H_y}.$$

For every $d \leq 64$ and every prime $q \mid H_y$ one has $q \nmid 2d$. In Moree's Euler-product formula this is exactly the perfect-power factor $q(q-2)/(q^2-q-1)$; see [22, Eq. (8)], specifically the factor attached to $q \mid h$ and $q \nmid mt$. Consequently

$$\mathfrak{M}_{b_y}(64) = \mathfrak{M}_2(64) \prod_{\substack{67 \leq q \leq y \\ q \text{ prime}}} \frac{q(q-2)}{q^2-q-1}.$$

The factors are $1 - 1/q + O(q^{-2})$, so the product tends to zero. This proves (iv). $\square$

Remark 12.6 (A fixed-base prime-side comparison through 64). *Under GRH, the same interval $1 \leq d \leq 64$ which contains only*

$$C^*(64) = 231$$

distinct odd composite exceptions in Somer's nontrivial-base convention contains at least

$$98.2709160681865 \dots \%$$

of the fixed-base prime mass for every positive non-perfect-power integer base. For correction-free Wagstaff behavior the corresponding prime mass is

$$98.4529145953885 \dots \%.$$

These percentages are fixed-base prime densities; $C^(64)$ is an existential-over-bases composite cardinality. The display is therefore a side-by-side structural comparison, not a probability theorem.*

Remark 12.7. *The prime-mass theorem clarifies two different uses of the word “generic”. For correction-free Wagstaff behavior the mass through 64 is $98.452915 \dots \%$. Over the larger odd-index generic class the worst case is the squarefree part 2, giving the sharp $98.270916 \dots \%$. Over all bases, including arbitrarily high perfect powers, the infimum is 0. Thus the correction-free value $98.452915 \dots \%$ is not a universal lower bound over all non-perfect-power bases, while the sharp universal value $98.270916 \dots \%$ is.*

For example,

$$E(5) = \frac{A}{25} \frac{24}{19} \approx 0.0189,$$

so approximately 1.89% of all primes lie on the fixed-base spectral line $d = 5$ for every odd-index generic base.

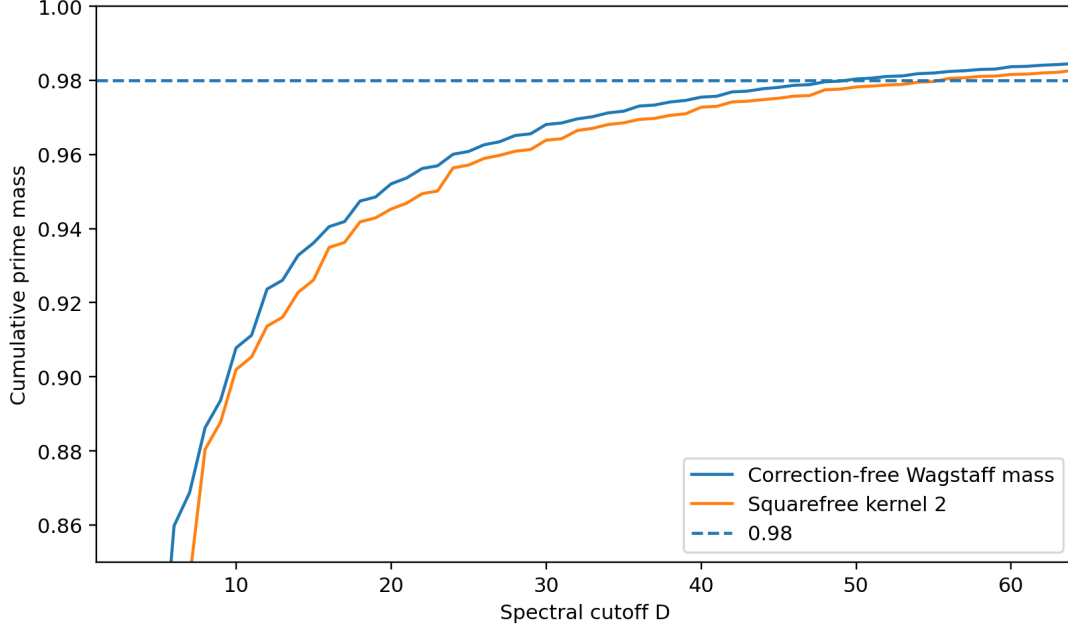


Figure 2: Cumulative fixed-base prime mass through spectral index D . The upper curve is the correction-free Wagstaff mass $\sum_{d \leq D} E(d)$. The lower curve is $\mathfrak{M}_2(D)$, the cumulative mass for squarefree kernel 2; its endpoint at $D = 64$ is the proven global minimum among positive non-perfect-power bases. No claim is made here that kernel 2 minimizes every intermediate cutoff D . The dashed horizontal line, labeled 0.98, marks 98%.

The distinction between the two prime spectra should be kept explicit:

$$\underbrace{\frac{1}{\phi(d)}}_{\text{existential over bases}} \quad \text{versus} \quad \underbrace{\delta(b, d)}_{\text{one fixed base}}.$$

The first quantity is unconditional and is the correct comparison with the union-over-bases composite spectrum $\mathcal{E}(d)$. The second is a refined, fixed-base Artin spectrum and is conditional on GRH in the present state of the theory.

13 The range beyond $d = 64$

The certified theorem in this paper now stops at $d = 64$. No exploratory table for larger d is used as evidence or as an input to any proof. Extending the exact branch search to $34 \leq d \leq 128$ is a natural next computation, with the next dyadic transition occurring at $d = 65$.

14 Further structural refinements

The overlap recurrence (18), introduced above as a certified search invariant, also has a useful structural interpretation. With

$$c_i = p_i^{a_i-1}(p_i - 1), \quad \Lambda_k = \text{lcm}(c_1, \dots, c_k), \quad H_k = \frac{\prod_{i=1}^k c_i}{\Lambda_k},$$

one has $H_s = \phi(n)/\lambda(n)$, so every prefix consumes part of a finite “overlap budget”. In the minimal-overlap case

$$H_s = 2^{s-1},$$

one obtains

$$\gcd(c_i, c_j) = 2 \quad (i \neq j).$$

For squarefree n this reduces to

$$\gcd(p_i - 1, p_j - 1) = 2.$$

The terminal prime is also strongly constrained. If $p_s > \kappa$, then

$$p_s - 1 \mid \kappa(P_{s-1} - 1).$$

Together with (18), this turns final-prime completion into a finite divisor problem. These refinements are central to the ongoing extension beyond $d = 64$.

15 Discussion and open problems

The spectral formulation suggests several problems.

1. Determine the exact growth of

$$N_{\max}(d) = \max \mathcal{E}(d),$$

with the convention $N_{\max}(d) = 0$ when $\mathcal{E}(d) = \emptyset$. The explicit function $U_0(d)$ of Theorem 7.8 proves finiteness but is far from optimal.

2. Determine the asymptotic behavior of

$$e(d) = |\mathcal{E}(d)|.$$

The data for $d \leq 64$ show strong fluctuations and repeated inherited spectral lines.

3. Determine the asymptotic behavior of the genuine count

$$g(d) = |\mathcal{G}(d)|$$

and compare it with the inclusive incidence count $e(d)$. By Corollary 3.3, $g(d)$ is the distribution of the canonical genuine index across all odd composites.

4. Extend the certified classification beyond $d = 64$. The next dyadic threshold is 65, followed by 129, $\dots$; these are natural complexity transitions because another distinct prime factor becomes possible.
5. Develop the ϕ -spectrum. Its finiteness is substantially more delicate and contains Lehmer-type problems.
6. For fixed base b , determine and compare the prime density $\delta_b(d)$ and the fixed-base pseudoprime count

$$|\mathcal{E}_b(d)|.$$

This is the most direct fixed-base version of the spectral-separation program.

15.1 Four natural OEIS sequences

The certified rows now define substantially longer prefixes of four natural integer sequences. A search of the current OEIS finds many fixed-base pseudoprime, Carmichael and fixed-order sequences, but we have not located a direct entry for Somer's variable-base Fermat d -pseudoprime spectrum in the inclusive form used here. For example, OEIS A086249 counts *base-2* Fermat pseudoprimes by the value of $\text{ord}_n(2)$; that fixed-base, fixed-order statistic differs from the variable-base reciprocal index studied here. The following prefixes are now certified through $d = 64$ with no numerical cutoff on n .

Let

$$\mathcal{G}(d) = \{n : d_\lambda(n) = d\}.$$

The four sequences are as follows.

1. **Inclusive spectral list, retaining inherited repetitions.** Read the rows $\mathcal{E}(1), \mathcal{E}(2), \dots$ in increasing d , sorting each row increasingly. Through $d = 64$ the complete certified prefix is

9, 6601, 25, 15, 561, 9, 49, 30889, 88561, 534061, 21, 6601, 45, 25, 121, 27, 126673, 1333333, 15, 169, 561, 91, 6601, 33, 49, 65, 35, 139231, 289, 30889, 88561, 534061, 55462177, 8885251441, 42018333841, 39, 153, 4371, 7107, 21, 361, 3201, 6601, 85, 8695, 10585, 26335, 9717247, 16666651, 37769887, 45, 133, 2465, 231, 645, 1105, 24013, 34133, 25, 49, 121, 529, 51, 6601, 15051, 11921001, 27, 105, 126673, 1333333, 55, 325, 3565, 30889, 88561, 534061, 57, 169, 561, 117, 175, 1045, 1447681, 2121061, 164424781, 91, 841, 6601, 158401, 63, 125, 435, 62745, 1154131, 2609581, 3936691, 5624641, 33, 65, 385, 961, 12673, 12801, 1047619, 2210671, 3494701, 15197491, 25158871, 93755971, 35, 69, 341, 139231, 561, 91001, 610051, 683761, 145, 217, 289, 30889, 88561, 534061, 55462177, 8885251441, 42018333841, 75, 17169, 33153, 45141, 62605, 2415953, 2929291, 11972017, 16894571, 67902031, 191834567, 39, 77, 153, 1065, 1369, 4371, 7107, 13833, 1170401, 703, 15211, 126673, 334153, 1333333, 81, 361, 481, 3201, 6601, 460801, 1372801, 165, 247, 2871, 8365, 73555, 184255, 9345541, 128950249, 85, 169, 1681, 8695, 10585, 26335, 9717247, 16666651, 37769887, 87, 259, 861, 22791, 52633, 69231, 82733, 45, 133, 1849, 2465, 15841, 91, 451, 8911, 88561, 534061, 3763801, 14245741, 93, 185, 231, 369, 645, 1105, 24013, 34133, 95, 1035, 2821, 4795, 204263, 223721, 308039, 559489, 38649041, 49, 529, 1729, 2209, 52417, 99, 87676681, 133861729, 161005573, 258844951, 1411040359, 3462426241, 7950140448251, 131097700645913, 798178197617711, 989473937059427, 1208361237478669, 51, 301, 1001, 6601, 15051, 3679201, 11921001, 4199932801, 1353385637801, 205, 32131, 139231, 105, 833, 9361, 126673, 741937, 1333333, 206955841, 560630848961, 425, 637, 3605, 8481, 55651, 3547291, 3984541, 43523707, 52519291, 260410837, 1271325841, 152674725071, 3916268927681, 55, 325, 2809, 3565, 30889, 88561, 534061, 4169867689, 111, 221, 1431, 6601, 15181, 100101, 689701, 1269621, 271794601, 568898551, 57, 169, 225, 561, 25761, 376993, 115, 343, 2737, 95761, 37883341, 713798461, 117, 175, 1045, 2553, 80041, 1447681, 2121061, 8372881, 164424781, 119, 1653, 1771, 42353741, 121, 361, 841, 3481, 6601, 25201, 158401, 6840001, 123, 245, 1221, 44409, 50997, 74665, 631351, 964411, 1102759, 940724555, 1962804565, 2417562005, 63, 125, 435, 3721, 19345, 62745, 264121, 828073, 1154131, 2609581, 3936691, 5624641, 7697921, 1891, 8695, 10585, 26335, 9717247, 16666651, 37769887, 65, 129, 385, 961, 1281, 12673, 12801.

There are 351 displayed terms because inherited occurrences are retained.

2. **Genuine spectral list.** Read the rows $\mathcal{G}(1), \mathcal{G}(2), \dots$ in increasing d . By Corollary 3.3, no odd composite occurs twice in this sequence. Through $d = 64$ the complete certified prefix is

9, 6601, 25, 15, 561, 49, 30889, 88561, 534061, 21, 45, 121, 27, 126673, 1333333, 169, 91, 33, 65, 35, 139231, 289, 55462177, 8885251441, 42018333841, 39, 153, 4371, 7107, 361, 3201, 85, 8695, 10585, 26335, 9717247, 16666651, 37769887, 133, 2465, 231, 645, 1105, 24013, 34133, 529, 51, 15051, 11921001, 105, 55, 325, 3565, 57, 117,

175, 1045, 1447681, 2121061, 164424781, 841, 158401, 63, 125, 435, 62745, 1154131, 2609581, 3936691, 5624641, 385, 961, 12673, 12801, 1047619, 2210671, 3494701, 15197491, 25158871, 93755971, 69, 341, 91001, 610051, 683761, 145, 217, 75, 17169, 33153, 45141, 62605, 2415953, 2929291, 11972017, 16894571, 67902031, 191834567, 77, 1065, 1369, 13833, 1170401, 703, 15211, 334153, 81, 481, 460801, 1372801, 165, 247, 2871, 8365, 73555, 184255, 9345541, 128950249, 1681, 87, 259, 861, 22791, 52633, 69231, 82733, 1849, 15841, 451, 8911, 3763801, 14245741, 93, 185, 369, 95, 1035, 2821, 4795, 204263, 223721, 308039, 559489, 38649041, 1729, 2209, 52417, 99, 87676681, 133861729, 161005573, 258844951, 1411040359, 3462426241, 7950140448251, 131097700645913, 798178197617711, 989473937059427, 1208361237478669, 301, 1001, 3679201, 4199932801, 1353385637801, 205, 32131, 833, 9361, 741937, 206955841, 560630848961, 425, 637, 3605, 8481, 55651, 3547291, 3984541, 43523707, 52519291, 260410837, 1271325841, 152674725071, 3916268927681, 2809, 4169867689, 111, 221, 1431, 15181, 100101, 689701, 1269621, 271794601, 568898551, 225, 25761, 376993, 115, 343, 2737, 95761, 37883341, 713798461, 2553, 80041, 8372881, 119, 1653, 1771, 42353741, 3481, 25201, 6840001, 123, 245, 1221, 44409, 50997, 74665, 631351, 964411, 1102759, 940724555, 1962804565, 2417562005, 3721, 19345, 264121, 828073, 7697921, 1891, 129, 1281.

It contains the 234 distinct composites that occur in the inclusive spectrum through $d = 64$.

3. Inclusive spectral counts.

$$e(d) = |\mathcal{E}(d)|.$$

For $1 \leq d \leq 64$:

0, 0, 0, 1, 1, 1, 2, 2, 3, 2, 1, 2, 3, 3, 2, 3,
 2, 7, 4, 4, 7, 3, 5, 4, 4, 4, 6, 3, 6, 4, 8, 6,
 6, 4, 4, 9, 11, 9, 5, 7, 8, 9, 7, 5, 7, 8, 9, 5,
 12, 9, 3, 8, 13, 8, 10, 6, 6, 9, 4, 8, 12, 13, 7, 7

4. Genuine spectral counts.

$$g(d) = |\mathcal{G}(d)|.$$

For $1 \leq d \leq 64$:

0, 0, 0, 1, 1, 1, 2, 1, 3, 1, 1, 1, 3, 1, 1, 2,
 2, 4, 4, 2, 7, 2, 5, 1, 3, 1, 3, 1, 6, 2, 8, 4,
 6, 2, 3, 2, 11, 5, 3, 4, 8, 1, 7, 2, 4, 3, 9, 3,
 12, 5, 2, 5, 13, 2, 9, 3, 6, 3, 4, 3, 12, 5, 1, 2

The two list sequences should be cross-referenced to the two count sequences, since the counts recover the row boundaries, including the empty rows $d = 1, 2, 3$. All four use the *inclusive* spectrum. An OEIS submission should state explicitly that the witnessing base may vary with n , that inherited and genuine appearances are different notions, and that the inclusive convention allows the residue classes ± 1 , unlike Somer's nontrivial-base convention. The supplementary machine-readable `combined_d1_64.json` is the preferred source for preparing OEIS b-files, because it preserves the row boundaries and exact integer values.

16 Conclusion

The starting question of this extended version is not initially spectral: if an exact reciprocal period

$$L = \text{ord}_n(b)$$

is known, what can it tell us about primality? The answer is a collection of classical and newer tools. At one extreme, $L = n - 1$ gives Lucas' immediate primality criterion. More generally, the exact order supplies Fermat and strong-pseudoprime information, Midy decompositions, cyclotomic structure, and sometimes enough of $n - 1$ to complete a Pocklington or Brillhart–Lehmer–Selfridge certificate. The toolbox also makes clear its own limitation: obtaining an exact order can be substantial computational input.

The spectral investigation begins when $L \mid n - 1$ and

$$d = \frac{n - 1}{L}.$$

It then gives two direct extensions of the Lucas idea. The exact theorem through $d = 64$ says that an odd n with $L = (n - 1)/d$ is prime unless it belongs to one explicitly certified row of Table 2; the 64 rows contain 234 distinct composite integers in 351 spectral incidences. The uniform theorem says that every composite spectral point satisfies

$$n < (dS(d))^{2M(d)},$$

and therefore sufficiently small d relative to $\log n / \log \log n$ forces primality without any finite table.

The genuine spectrum adds a complementary base-independent structure. Every odd composite has exactly one genuine index

$$d_\lambda(n) = \frac{n - 1}{\gcd(n - 1, \lambda(n))},$$

so the sets $\mathcal{G}(d)$ partition all odd composites. The counting problem is now more quantitative than in the preceding drafts. The λ -overlap search has $B(d) = \frac{1}{2}d(\log d)^2 + O(d \log d)$ parameter branches, while a relaxed collapsed master-equation count has only $B_{\text{mast}}^+(d) = d \log d + O(d)$ candidate pairs. Solving the last two primes by a divisor-pair identity gives

$$\log e(d) \leq \frac{d}{2}(\log d + \log \log d + O(1)).$$

Under the natural line-sampling model this implies not only that the conditional primality probability tends to 1 for every fixed d , but uniformly, for each fixed $0 < \varepsilon < 2$, throughout

$$d \leq (2 - \varepsilon) \frac{\log X}{\log \log X}.$$

Thus the deterministic certainty threshold with leading constant $1/2$ and the high-probability threshold with leading constant 2 give complementary ways in which a long reciprocal period signals primality.

The prime side exhibits a different kind of concentration. Under GRH, the fixed-base Wagstaff–Moree theory places at least 98.2709160681865... % of the prime mass in $d \leq 64$ for every positive integer base that is not a perfect power; the correction-free mass is 98.4529145953885... %. Thus the same low-index interval contains most of the prime mass but only a very small certified composite population.

Historically, the reciprocal index was developed here from the period–primality question before Somer's 1985 dissertation and 1989 paper were located through an AI-assisted literature search. Somer's earlier work establishes the correct priority for fixed- d finiteness and the $d \leq 8$ classification. A targeted citation search shows real but relatively narrow independent uptake, most visibly in Ribenboim's dedicated "Somer Pseudoprimes" subsection and Pinch's bounded-index Carmichael work. We therefore regard the Fermat d -pseudoprime theory as under-recognized rather than forgotten. The independent rediscovery from a different motivation nevertheless suggests that $(n - 1)/L$ is a natural arithmetic invariant. The main questions now are quantitative: determine the true growth of $e(d)$ and $g(d)$, sharpen the uniform primality boundary, extend the exact spectrum beyond 33, and understand how the L -, λ -, ϕ -, and fixed-base Artin spectra fit together.

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Declaration on computational and AI assistance

During the development of this work, generative-AI tools, principally OpenAI ChatGPT, were used as interactive aids for exploratory algebra, code drafting, literature discovery, and manuscript editing. In particular, an AI-assisted literature search first brought Somer’s 1985 dissertation and 1989 Fermat d -pseudoprime paper to the authors’ attention after the reciprocal-index viewpoint and initial small-index arguments had already been developed independently. This historical sequence is stated to explain the development of the project, not to claim priority over Somer’s results.

All literature claims used mathematically are tied to human-authored sources. All computer-assisted classification claims are tied to the frozen Julia source, captured execution logs, and separate arithmetic verifier described in Subsection 9.3. The human authors retain full responsibility for the mathematical statements, citations, and the verification of every theorem and computational claim in this manuscript.

References

- [1] L. M. Adleman, C. Pomerance, and R. S. Rumely, *On distinguishing prime numbers from composite numbers*, Ann. of Math. (2) **117** (1983), 173–206.
- [2] M. Agrawal, N. Kayal, and N. Saxena, *PRIMES is in P*, Ann. of Math. **160** (2004), 781–793.
- [3] W. R. Alford, A. Granville, and C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. of Math. **140** (1994), 703–722.
- [4] A. O. L. Atkin and F. Morain, *Elliptic curves and primality proving*, Math. Comp. **61** (1993), 29–68, doi:10.1090/S0025-5718-1993-1199989-X.
- [5] J. Brillhart, D. H. Lehmer, and J. L. Selfridge, *New primality criteria and factorizations of $2^m \pm 1$* , Math. Comp. **29** (1975), 620–647.
- [6] W. Carlip, E. Jacobson, and L. Somer, *Pseudoprimes, perfect numbers, and a problem of Lehmer*, Fibonacci Quart. **36** (1998), 361–371.
- [7] W. Carlip and L. Somer, *Square-free Lucas d -pseudoprimes and Carmichael–Lucas numbers*, Czechoslovak Math. J. **57** (2007), 447–463.
- [8] J. H. Castillo, G. García-Pulgarín, and J. M. Velásquez-Soto, *Pseudoprimes stronger than strong pseudoprimes*, arXiv:1202.3428, 2012.
- [9] J. H. Castillo, G. García-Pulgarín, and J. M. Velásquez-Soto, *De los números de Midy a la primalidad*, Rev. Integr. Temas Mat. **33** (2015), 1–10, doi:10.18273/revint.v33n1-2015001.
- [10] H. Cohen and H. W. Lenstra, Jr., *Primality testing and Jacobi sums*, Math. Comp. **42** (1984), 297–330.
- [11] R. Crandall and C. Pomerance, *Prime Numbers: A Computational Perspective*, 2nd ed., Springer, 2005.
- [12] G. García-Pulgarín and H. Giraldo, *Characterizations of Midy’s property*, Integers **9** (2009), 191–197.
- [13] A. Gupta and B. Sury, *Decimal expansion of $1/p$ and subgroup sums*, Integers **5** (2005), A19.
- [14] C. Hooley, *On Artin’s conjecture*, J. Reine Angew. Math. **225** (1967), 209–220.

- [15] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, Vol. 53, AMS, Providence, RI, 2004.
- [16] M. Kamada, *Factorizations of $\Phi_n(10)$* , STUDIO KAMADA, continuously updated online factor tables,
<https://stdkmd.net/nrr/repunit/phin10.htm>.
- [17] M. Kamada, *Probable prime factors of repunit*, STUDIO KAMADA, continuously updated online appendix,
<https://stdkmd.net/nrr/repunit/prpfactors.htm>.
- [18] D. H. Lehmer, *On Euler's totient function*, Bull. Amer. Math. Soc. 38 (1932), 745–751.
- [19] J. Lewittes, *Midy's theorem for periodic decimals*, Integers 7 (2007), A02.
- [20] H. W. Martin, *Generalizations of Midy's theorem on repeating decimals*, Integers 7 (2007), A03.
- [21] P. Moree, *Asymptotically exact heuristics for (near) primitive roots*, J. Number Theory 83 (2000), 155–181.
- [22] P. Moree, *Near-primitive roots*, Funct. Approx. Comment. Math. 48 (2013), 133–145, doi:10.7169/facm/2013.48.1.11.
- [23] R. G. E. Pinch, *The Carmichael numbers up to 10^{18}* , arXiv:math/0604376, 2006.
- [24] R. G. E. Pinch, *The Carmichael numbers up to 10^{21}* , in A.-M. Ernvall-Hytönen, M. Jutila, J. Karhumäki, and A. Lepistö (eds.), *Proceedings of the Conference on Algorithmic Number Theory*, TUCS General Publication 46, Turku Centre for Computer Science, 2007, pp. 129–131.
- [25] R. G. E. Pinch, *Carmichael numbers with small index*, ANTS VII poster, 2006, https://antsmath.org/ANTSVII/Poster/Pinch_Poster2.pdf.
- [26] C. Pomerance, *On composite n for which $\phi(n) \mid n - 1$* , Acta Arith. 28 (1976), 387–389, doi:10.4064/aa-28-4-387-389.
- [27] C. Pomerance, *A tale of two sieves*, Notices Amer. Math. Soc. 43 (1996), 1473–1485.
- [28] V. R. Pratt, *Every prime has a succinct certificate*, SIAM J. Comput. 4 (1975), 214–220.
- [29] P. Ribenboim, *The New Book of Prime Number Records*, Springer, New York, 1996.
- [30] V. Shevelev, G. García-Pulgarín, J. M. Velásquez-Soto, and J. H. Castillo, *Overpseudoprimes, and Mersenne and Fermat numbers as primover numbers*, J. Integer Seq. 15 (2012), Article 12.7.7.
- [31] P. W. Shor, *Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer*, SIAM J. Comput. 26 (1997), 1484–1509.
- [32] L. E. Somer, *The Divisibility and Modular Properties of K th-Order Linear Recurrences over the Ring of Integers of an Algebraic Number Field with Respect to Prime Ideals*, Ph.D. dissertation, University of Illinois at Urbana–Champaign, 1985.
- [33] L. Somer, *On Fermat d -pseudoprimes*, in J.-M. De Koninck and C. Levesque (eds.), *Number Theory: Proceedings of the International Number Theory Conference, Université Laval, 1987*, de Gruyter, 1989, pp. 841–860, doi:10.1515/9783110852790.841.

- [34] L. Somer, *On Lucas d -pseudoprimes*, in G. E. Bergum, A. N. Philippou, and A. F. Horadam (eds.), *Applications of Fibonacci Numbers*, Vol. 7, Kluwer Academic Publishers, 1998, pp. 369–375.
- [35] J. P. Sorenson and J. Webster, *Strong pseudoprimes to twelve prime bases*, Math. Comp. **86** (2017), 985–1003, doi:10.1090/mcom/3134.
- [36] A. V. Sutherland, *Order Computations in Generic Groups*, Ph.D. thesis, Massachusetts Institute of Technology, 2007.
- [37] S. S. Wagstaff, Jr., *Pseudoprimes and a generalization of Artin’s conjecture*, Acta Arith. 41 (1982), 141–150, doi:10.4064/aa-41-2-141-150.