Power-k Equal-Detour Points of Triangles and Simplices

Existence, Closure, and the Soddy–Orthocenter Specializations

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Abstract

For a point P and two vertices A_i, A_j, the ordinary detour through P is

$$PA_i + PA_j - A_iA_j.$$

This note replaces all three distances by their k-th powers and studies points for which the resulting quantity is independent of the chosen edge. The resulting *power-k equal-detour points* put two familiar geometries into one scheme. For $k = 1$ one recovers the classical equal-detour/Soddy geometry. For $k = 2$ one recovers the orthocenter, and in higher dimension the natural carrier is the family of orthocentric simplices. The key algebraic object is the power-additive edge law

$$A_iA_j^k = u_i + u_j.$$

Every triangle satisfies such a law for every $k > 0$, while for tetrahedra it imposes two independent conditions and therefore defines a four-dimensional family in the six-dimensional edge-length space. We derive an explicit barycentric curve and a scalar Gram-matrix closure equation. For every $k > 1$, power-additivity is both necessary and sufficient for existence, so every triangle has an equal-detour point in this range. We also establish local analytic continuation through the orthocenter and determine all possible universal positive exponents for the three classical candidates $X(4)$, $X(175)$ and $X(176)$. A separate, explicitly qualified finite-catalog computation is reported as evidence rather than used as a proof of the geometric results.

Keywords. equal detour; simplex; orthocenter; Soddy circle; power-additive edges; interval arithmetic.

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1 Motivation and preliminaries

Let

$$S = [A_0, \dots, A_n] \subset \mathbb{R}^n$$

be a nondegenerate Euclidean n-simplex, with $n \geq 2$. Write

$$d_{ij} = |A_i - A_j|$$

for its edge lengths. Throughout, $k > 0$ is a fixed real number.

For $k = 1$, the extra distance incurred by travelling from A_i to A_j through a point P instead of directly along the edge is

$$|P - A_i| + |P - A_j| - d_{ij}.$$

In a triangle, requiring the same extra distance for all three edges gives the classical equal-detour point $X(176)$, and in certain triangles a second equal-detour point also occurs; these points are naturally described by the Soddy circles [2, 1].

The orthocenter suggests replacing distance by squared distance. If H is the orthocenter of a triangle or, more generally, of an orthocentric simplex, then the off-diagonal scalar products

$$(A_i - H) \cdot (A_j - H), \qquad i \neq j,$$

are all equal [3]. Since

$$|H - A_i|^2 + |H - A_j|^2 - d_{ij}^2 = 2(A_i - H) \cdot (A_j - H),$$

the orthocenter is an "equal squared-detour" point. This motivates the following definition.

2 Power-k equal detours

Definition 1 (Power-k detour)**.** *For an edge A_iA_j and a point $P \in \mathbb{R}^n$, define*

$$D_{ij}^{(k)}(P) := |P - A_i|^k + |P - A_j|^k - d_{ij}^k. \tag{1}$$

Definition 2 (Power-k equal-detour point)**.** *A point P is a* power-k equal-detour point *of S if there is a constant K such that*

$$D_{ij}^{(k)}(P) = K \qquad \text{for every } i \neq j. \tag{2}$$

The set of all such points is denoted by

$$\mathcal{E}_k(S).$$

It is important that $\mathcal{E}_k(S)$ is a set: it need not consist of a single point.

The first structural consequence of the definition is immediate but fundamental.

Proposition 1 (Power-additive edge law)**.** *If $P \in \mathcal{E}_k(S)$ and the common detour value is K, then*

$$d_{ij}^k = u_i + u_j, \qquad u_i := |P - A_i|^k - \frac{K}{2}. \tag{3}$$

Conversely, suppose the edge powers admit a representation

$$d_{ij}^k = u_i + u_j.$$

Then P is a power-k equal-detour point if and only if

$$|P - A_i|^k - u_i = \lambda \qquad (i = 0, \ldots, n) \tag{4}$$

for one common number λ. In that case $K = 2\lambda$.

Proof. Equation (2) can be rearranged as

$$d_{ij}^k = \left(|P - A_i|^k - \frac{K}{2}\right) + \left(|P - A_j|^k - \frac{K}{2}\right),$$

which gives the first statement. Conversely, if $d_{ij}^k = u_i + u_j$ and (4) holds, then

$$D_{ij}^{(k)}(P) = (u_i + \lambda) + (u_j + \lambda) - (u_i + u_j) = 2\lambda.$$

□

This motivates a family of simplices which is independent of the point P.

Definition 3 (Power-additive simplex)**.** *An n-simplex belongs to the* power-k additive family $\mathcal{P}_k$ *if there exist real numbers $u_0, \dots, u_n$ such that*

$$d_{ij}^k = u_i + u_j \qquad (i \neq j). \tag{5}$$

Thus

$$\boxed{\mathcal{E}_k(S) \neq \varnothing \quad \Longrightarrow \quad S \in \mathcal{P}_k.}$$

Membership in $\mathcal{P}_k$ separates metric compatibility from the closure problem. Theorem 3 below proves that it is also sufficient for every $k > 1$. The range $0 < k < 1$ requires a separate analysis; no counterexample to sufficiency is asserted here.

3 How large is the power-additive family?

Proposition 2 (Exact dimension)**.** *For fixed $k > 0$, the power-additive family $\mathcal{P}_k$ of Euclidean n-simplices is a nonempty smooth submanifold of positive edge-length space of dimension $n+1$ and codimension $(n+1)(n-2)/2$.*

Proof. The linear map $L(u) = (u_i + u_j)_{i<j}$ is injective: for distinct i, j, ℓ, its coordinates determine $2u_i = (u_i + u_j) + (u_i + u_\ell) - (u_j + u_\ell)$. The map $(d_{ij}) \mapsto (d_{ij}^k)$ is a diffeomorphism of the positive orthant. Intersecting $L(\mathbb{R}^{n+1})$ with the open set corresponding to nondegenerate Euclidean simplices therefore gives the claimed submanifold. It is nonempty because a regular simplex belongs to it. □

Thus the exact codimension is

$$\binom{n+1}{2} - (n+1) = \frac{(n+1)(n-2)}{2}. \tag{6}$$

In particular,

dimension	codimension of $\mathcal{P}_k$	dimension in edge space
2	0	3
3	2	4
4	5	5

before quotienting by overall scale. After that quotient the family has dimension n.

For four distinct vertices i, j, ℓ, m, additivity implies

$$d_{ij}^k + d_{\ell m}^k = d_{i\ell}^k + d_{jm}^k = d_{im}^k + d_{j\ell}^k. \tag{7}$$

Conversely, the collection of four-point relations (7) is precisely the consistency condition for representing all off-diagonal edge powers as $u_i + u_j$.

For a tetrahedron this reduces to the two independent equations

$$AB^k + CD^k = AC^k + BD^k = AD^k + BC^k. \tag{8}$$

Hence the $k = 1$ and $k = 2$ tetrahedral families are both four-dimensional families inside the six-dimensional space of edge lengths.

4 Triangles: every k is edge-additive

Let ABC be a triangle with the standard notation

$$a = BC, \qquad b = CA, \qquad c = AB,$$

and area Δ. There are only three edge equations and three vertex parameters, so for every $k > 0$ the power potentials are uniquely determined:

$$\boxed{\begin{aligned} u_A &= \frac{b^k + c^k - a^k}{2}, \\ u_B &= \frac{c^k + a^k - b^k}{2}, \\ u_C &= \frac{a^k + b^k - c^k}{2}. \end{aligned}} \tag{9}$$

Therefore

$$\boxed{\text{every nondegenerate triangle belongs to } \mathcal{P}_k \text{ for every } k > 0.}$$

Corollary 1 (Direct triangle test)**.** *For every $k > 0$,*

$$P \in \mathcal{E}_k(ABC) \quad \Longleftrightarrow \quad PA^k + a^k = PB^k + b^k = PC^k + c^k.$$

Proof. Use $u_A = (a^k + b^k + c^k)/2 - a^k$ and its cyclic analogues in (4). ☐

This does *not* mean that an equal-detour point is automatically unique. For a general k, existence and multiplicity are controlled by one scalar closure equation, which we now derive.

5 The barycentric functional curve for a triangle

Suppose that P is to satisfy

$$PA^k = u_A + \lambda, \qquad PB^k = u_B + \lambda, \qquad PC^k = u_C + \lambda.$$

Necessarily

$$\lambda \geq -\min\{u_A, u_B, u_C\}. \tag{10}$$

Define the desired squared distances

$$\delta_A(\lambda) = (u_A + \lambda)^{2/k}, \quad \delta_B(\lambda) = (u_B + \lambda)^{2/k}, \quad \delta_C(\lambda) = (u_C + \lambda)^{2/k}. \tag{11}$$

A point in the plane is determined affinely by the differences of its squared distances from the vertices. This gives a particularly useful barycentric parametrization. Define

$$\begin{aligned} W_A(\lambda) = a^2(b^2 + c^2 - a^2) - 2a^2\delta_A \\ + (a^2 + b^2 - c^2)\delta_B + (a^2 + c^2 - b^2)\delta_C, \end{aligned} \tag{12}$$

$$\begin{aligned} W_B(\lambda) = b^2(c^2 + a^2 - b^2) - 2b^2\delta_B \\ + (b^2 + c^2 - a^2)\delta_C + (b^2 + a^2 - c^2)\delta_A, \end{aligned} \tag{13}$$

$$\begin{aligned} W_C(\lambda) = c^2(a^2 + b^2 - c^2) - 2c^2\delta_C \\ + (c^2 + a^2 - b^2)\delta_A + (c^2 + b^2 - a^2)\delta_B. \end{aligned} \tag{14}$$

A direct calculation gives the useful identity

$$W_A + W_B + W_C = 16\Delta^2. \tag{15}$$

Consequently

$$\boxed{\Gamma_k(\lambda) = \left(\frac{W_A(\lambda)}{16\Delta^2}, \frac{W_B(\lambda)}{16\Delta^2}, \frac{W_C(\lambda)}{16\Delta^2}\right)} \tag{16}$$

is a normalized barycentric point for every admissible λ.

We call Γ_k the *barycentric functional curve*. It is important to distinguish the curve from the actual equal-detour set: a generic point of the curve has the correct *differences* of squared distances, but its absolute distances need not yet be the values prescribed in (11).

To impose the remaining condition, write

$$\alpha(\lambda) = \frac{W_A}{16\Delta^2}, \qquad \beta(\lambda) = \frac{W_B}{16\Delta^2}, \qquad \gamma(\lambda) = \frac{W_C}{16\Delta^2}.$$

With A as origin, the vector $P - A$ equals

$$\beta(B - A) + \gamma(C - A),$$

so

$$\boxed{F_k(\lambda) := \beta^2 c^2 + \gamma^2 b^2 + \beta\gamma(b^2 + c^2 - a^2) - \delta_A(\lambda) = 0.} \tag{17}$$

Theorem 1 (Triangle closure principle)**.** *For a nondegenerate triangle ABC and $k > 0$, the finite power-k equal-detour points are exactly*

$$\boxed{\mathcal{E}_k(ABC) = \{\Gamma_k(\lambda) : \lambda \geq -\min u_i,\ F_k(\lambda) = 0\}.} \tag{18}$$

Repeated roots and endpoint roots are allowed. Thus existence and multiplicity reduce to a one-variable problem.

The formula also has the usual triangle-center symmetry: W_A is unchanged when B and C are interchanged, provided the corresponding quantities are interchanged. A symmetric branch defines a similarity-equivariant center when it also satisfies $\lambda(ta, tb, tc) = t^k \lambda(a, b, c)$ for $t > 0$. The W_i then scale uniformly by t^4. Its domain and its continuation at root collisions must be specified; symmetry alone does not supply a global branch choice.

6 The case $k = 1$: Soddy and equal-detour geometry

For $k = 1$, (9) becomes

$$u_A = s - a, \qquad u_B = s - b, \qquad u_C = s - c, \tag{19}$$

where $s = (a + b + c)/2$. These are positive by the triangle inequalities. They are exactly the radii of the three mutually externally tangent circles centered at A, B, C:

$$AB = u_A + u_B, \quad BC = u_B + u_C, \quad CA = u_C + u_A.$$

If a fourth circle of positive radius λ is externally tangent to all three, then its center P satisfies

$$PA = u_A + \lambda,$$

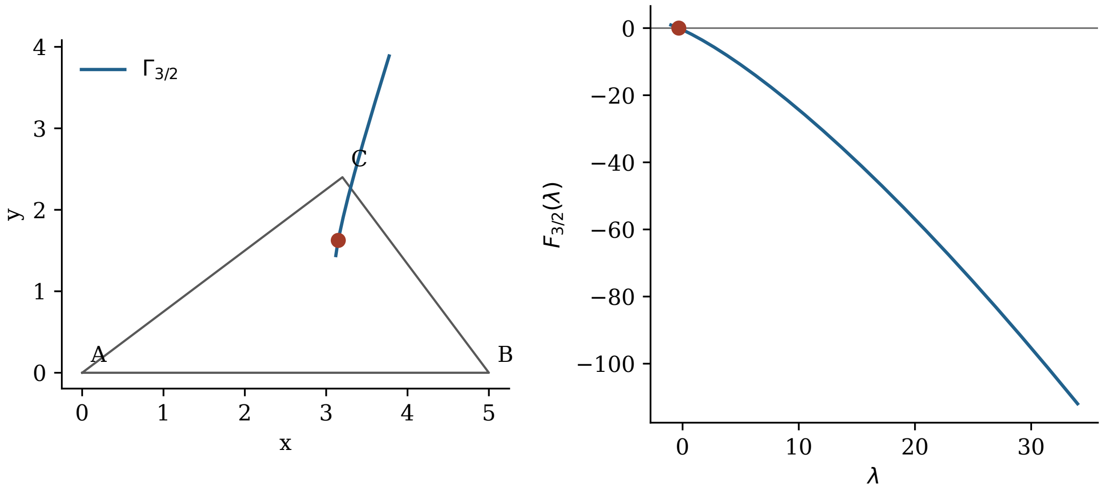

Figure 1: Computed example for the 3–4–5 triangle and $k = 3/2$. The left panel shows the auxiliary curve Γ_k over a finite parameter interval; the right panel shows the closure function. A marked root gives an actual equal-detour point. Unmarked curve points satisfy only the distance-difference equations. The figure illustrates a computed root and makes no global uniqueness claim.

and cyclically. Hence

$$D^{(1)}_{AB}(P) = 2\lambda,$$

and similarly for all three edges. This is precisely the classical equal-detour construction. Hajja and Yff give a clean treatment of its relation to the two Soddy circles and correct the older existence/uniqueness conditions [2].

Let

$$\kappa_A = \frac{1}{s-a}, \qquad \kappa_B = \frac{1}{s-b}, \qquad \kappa_C = \frac{1}{s-c}.$$

Descartes' circle theorem gives the two oriented completion curvatures

$$\kappa_\pm = \kappa_A + \kappa_B + \kappa_C \pm 2\sqrt{\kappa_A\kappa_B + \kappa_B\kappa_C + \kappa_C\kappa_A}. \tag{20}$$

The positive "+" root always yields the inner Soddy circle and the equal-detour point $X(176)$. Its barycentrics simplify to

$$\boxed{X(176) = \left(a + \frac{\Delta}{s-a} : b + \frac{\Delta}{s-b} : c + \frac{\Delta}{s-c}\right).} \tag{21}$$

The classical companion center is

$$\boxed{X(175) = \left(a - \frac{\Delta}{s-a} : b - \frac{\Delta}{s-b} : c - \frac{\Delta}{s-c}\right),} \tag{22}$$

the center of the outer Soddy circle. When the lower Descartes curvature κ_- is positive, this outer circle is also externally tangent, and $X(175)$ is a second point of equal detour. When $\kappa_- < 0$, the outer circle encloses the three vertex circles; its center instead has the isoperimetric interpretation. At $\kappa_- = 0$ the outer circle becomes a line.

The sign change can be written in familiar triangle quantities as

$$\kappa_- \gtrless 0 \quad\Longleftrightarrow\quad 4R + r \gtrless 2s, \tag{23}$$

where R and r are the circumradius and inradius. Thus $X(176)$ is always an equal-detour point, while $X(175)$ is a second equal-detour point exactly on the side $4R + r > 2s$ of the Soddy transition. This agrees with the corrected uniqueness analysis of Hajja–Yff [2].

7 The case $k = 2$: the orthocenter

For $k = 2$,

$$u_A = \frac{b^2 + c^2 - a^2}{2}, \quad u_B = \frac{c^2 + a^2 - b^2}{2}, \quad u_C = \frac{a^2 + b^2 - c^2}{2}. \tag{24}$$

Now

$$\delta_i = u_i + \lambda,$$

so all differences $\delta_i - \delta_j$ are independent of λ. Consequently the barycentric functional curve *collapses to one point.* Equations (12)–(14) reduce to

$$\boxed{\begin{aligned} W_A &= (a^2 + b^2 - c^2)(a^2 + c^2 - b^2), \\ W_B &= (b^2 + c^2 - a^2)(b^2 + a^2 - c^2), \\ W_C &= (c^2 + a^2 - b^2)(c^2 + b^2 - a^2). \end{aligned}} \tag{25}$$

These are proportional to

$$\tan A : \tan B : \tan C,$$

the barycentric coordinates of the orthocenter H. Hence

$$\boxed{\mathcal{E}_2(ABC) = \{H\}.} \tag{26}$$

For a right triangle, the same formula is understood by the usual limiting interpretation and H is the right-angle vertex.

The geometric reason is especially simple:

$$HA_i^2 + HA_j^2 - A_iA_j^2 = 2(A_i - H) \cdot (A_j - H), \tag{27}$$

and the scalar products on the right are equal for all distinct pairs in an orthocentric system.

8 Higher dimensions: an exact Gram-matrix formulation

Let $S = [A_0, \ldots, A_n]$ be a nondegenerate n-simplex in $\mathbb{R}^n$ and suppose first that $S \in \mathcal{P}_k$, so

$$d_{ij}^k = u_i + u_j.$$

Choose A_0 as origin and let

$$e_i = A_i - A_0 \qquad (i = 1, \ldots, n).$$

The $n \times n$ Gram matrix is

$$G_{ij} = e_i \cdot e_j = \frac{d_{0i}^2 + d_{0j}^2 - d_{ij}^2}{2}. \tag{28}$$

It is positive definite because the simplex is nondegenerate.

For an admissible parameter λ, put

$$\delta_i(\lambda) = (u_i + \lambda)^{2/k} \qquad (i = 0, \ldots, n) \tag{29}$$

and define

$$v_i(\lambda) = \frac{d_{0i}^2 + \delta_0(\lambda) - \delta_i(\lambda)}{2}, \qquad i = 1, \dots, n. \tag{30}$$

If

$$P = A_0 + \sum_{i=1}^{n} \xi_i e_i,$$

then the squared-distance differences give simply

$$\boxed{\xi(\lambda) = G^{-1} v(\lambda).} \tag{31}$$

Thus the normalized barycentric functional curve is

$$\boxed{\beta_0(\lambda) = 1 - \sum_{i=1}^{n} \xi_i(\lambda), \qquad \beta_i(\lambda) = \xi_i(\lambda).} \tag{32}$$

Exactly as in the triangle, one scalar equation remains:

$$\boxed{\Phi_k(\lambda) := v(\lambda)^T G^{-1} v(\lambda) - \delta_0(\lambda) = 0.} \tag{33}$$

Theorem 2 (Simplex closure principle)**.** *Let $S \in \mathcal{P}_k$ be a nondegenerate Euclidean n-simplex. Then the finite power-k equal-detour points are in one-to-one correspondence with the admissible real roots of (33). For every such root, the barycentric coordinates are given by (31)–(32).*

Proof. The linear equations for squared-distance differences uniquely determine the auxiliary point. The remaining closure equation sets the squared distance to A_0 correctly; every other squared distance then agrees as well. Taking nonnegative powers recovers (4). Conversely, an actual equal-detour point determines a unique $\lambda = |P - A_0|^k - u_0$, proving the claimed bijection. □

8.1 Existence throughout the range $k > 1$

Theorem 3 (Existence for $k > 1$)**.** *For every fixed $k > 1$ and every nondegenerate Euclidean n-simplex S,*

$$\mathcal{E}_k(S) \neq \varnothing \iff S \in \mathcal{P}_k.$$

Proof. Necessity follows by rearranging the equal-detour equations. For sufficiency write $d_{ij}^k = u_i + u_j$, set $p = 2/k$, and define

$$\delta_i(\lambda) = (u_i + \lambda)^p, \qquad \lambda \geq \lambda_0 := -\min_i u_i.$$

Choose A_0 as origin, let G be the positive definite edge Gram matrix, and put

$$v_i(\lambda) = \frac{d_{0i}^2 + \delta_0(\lambda) - \delta_i(\lambda)}{2}, \qquad \Phi(\lambda) = v(\lambda)^T G^{-1} v(\lambda) - \delta_0(\lambda).$$

The auxiliary point $P(\lambda)$ with edge coordinates $G^{-1}v$ satisfies

$$|P(\lambda) - A_i|^2 - \delta_i(\lambda) = \Phi(\lambda) \quad \text{for every } i.$$

At λ_0, one δ_i is zero, so $\Phi(\lambda_0) \geq 0$. For large positive λ,

$$\delta_i = \lambda^p + p u_i \lambda^{p-1} + O(\lambda^{p-2}), \quad v = O(1 + \lambda^{p-1}).$$

Since $0 < p < 2$,

$$v^T G^{-1} v = O(1 + \lambda^{2p-2}) = o(\lambda^p), \qquad \Phi(\lambda)/\lambda^p \longrightarrow -1.$$

Continuity yields an admissible root. At a root all actual squared distances equal the nonnegative prescribed squared distances; taking their $k/2$ powers proves the required equations. □

Corollary 2. *Every nondegenerate triangle has a power-k equal-detour point for every $k > 1$.*

This conclusion asserts neither uniqueness nor interiority, and the asymptotic proof does not cover $k = 1$.

This formulation is useful because it cleanly separates two questions:

(i) *metric compatibility*: does the simplex belong to $\mathcal{P}_k$?;

(ii) *Euclidean closure*: does the scalar equation $\Phi_k(\lambda) = 0$ have an admissible root?

For $n = 2$, the first question is automatic. For $n \geq 3$, it is already restrictive.

9 $k = 1$ and $k = 2$ in higher dimensions

9.1 $k = 1$: balloons and Soddy–Gosset completion

For $k = 1$, the additive law is

$$d_{ij} = r_i + r_j. \tag{34}$$

For a genuine simplex the face triangle inequalities force $r_i > 0$. Geometrically, the vertices are therefore the centers of $n + 1$ mutually externally tangent n-balls. In dimension three these are the 4-ball, balloon, or edge-additive tetrahedra studied in the tetrahedral Soddy literature [4, 5, 6].

If a further hypersphere of positive radius λ is externally tangent to all $n + 1$ vertex balls, its center satisfies

$$|P - A_i| = r_i + \lambda,$$

so all ordinary detours equal 2λ. In oriented curvature notation $b_i = 1/r_i$, the two Soddy–Gosset completion curvatures satisfy

$$\left(\sum_{i=0}^{n+1} b_i\right)^2 = n \sum_{i=0}^{n+1} b_i^2. \tag{35}$$

If the $n+1$ base curvatures have sum B and sum of squares Q, then the two oriented completion curvatures are

$$\boxed{b_{\pm} = \frac{B \pm \sqrt{n(B^2 - (n-1)Q)}}{n-1}.} \tag{36}$$

A positive completion gives an ordinary equal-detour point. A negative completion represents an enclosing oriented hypersphere and should not be confused with an unsigned equal-detour point.

9.2 $k = 2$: orthocentric simplices

The power-2 additive law is

$$d_{ij}^2 = u_i + u_j. \tag{37}$$

This is a standard characterization of orthocentric simplices. Indeed, if H is the orthocenter and the origin is placed at H, then

$$A_i \cdot A_j = c \qquad (i \neq j)$$

for one constant c [3]. Therefore

$$d_{ij}^2 = |A_i|^2 + |A_j|^2 - 2c = (|A_i|^2 - c) + (|A_j|^2 - c).$$

Conversely, the additive squared-edge condition yields the orthocentric structure. Thus

$$\boxed{\mathcal{P}_2 = \{\text{orthocentric simplices}\}.} \tag{38}$$

The orthocenter is their power-2 equal-detour point in every dimension.

9.3 Local continuation through the orthocenter

Proposition 3 (Local branch through the orthocenter)**.** *For a fixed nonright triangle, there is a unique local real-analytic solution branch $P(k)$ through $P(2) = H$.*

Proof. Use the equations

$$|P - B|^k - |P - A|^k = u_B(k) - u_A(k), \quad |P - C|^k - |P - A|^k = u_C(k) - u_A(k).$$

At $(P, k) = (H, 2)$, the two Jacobian rows with respect to P are $2(A - B)^T$ and $2(A - C)^T$, and are independent. The distances from H to the vertices are nonzero for a nonright triangle, so the equations are analytic nearby. Apply the analytic implicit-function theorem. Uniqueness here is local in both P and k. □

9.4 Intersection of the tetrahedral specializations

Proposition 4. *A nondegenerate balloon tetrahedron with edge lengths $d_{ij} = r_i + r_j$, $r_i > 0$, is orthocentric if and only if at least three of the radii are equal.*

Proof. Differences of opposite squared-edge sums are twice the products

$$(r_0 - r_3)(r_1 - r_2), \quad (r_0 - r_2)(r_1 - r_3), \quad (r_0 - r_1)(r_2 - r_3),$$

up to the immaterial choice of signs. Three equal radii make all vanish. Conversely, if not all radii are equal, relabel so $r_0 \neq r_1$. The third product forces $r_2 = r_3 = t$, and either remaining equation gives $(r_0 - t)(r_1 - t) = 0$. Thus three radii equal t. □

10 The power-k picture

The preceding results suggest viewing the two classical geometries as neighboring members of one metric family:

k	Edge law	Natural simplex family	Equal-detour geometry
1	$d_{ij} = r_i + r_j$	balloon / edge-additive simplices	Soddy–Gosset, $X(176)$ in triangles
2	$d_{ij}^2 = u_i + u_j$	orthocentric simplices	orthocenter $H = X(4)$
general	$d_{ij}^k = u_i + u_j$	power-additive family $\mathcal{P}_k$	roots of $\Phi_k(\lambda) = 0$

Two observations are particularly important.

First, dimension two is exceptional. Since there are exactly three sides and exactly three vertex potentials, every triangle is power-additive for every k. In dimension three, the two opposite-edge conditions in (8) appear, so the unrestricted triangle phenomenon splits into special four-dimensional tetrahedral families.

Second, $\mathcal{P}_k$ should be regarded as the *natural carrier* of the power-k equal-detour problem, not automatically as its solution set. The actual point set is obtained only after the closure equation is imposed. Classical tangency supplies the $k = 1$ interpretation, and Theorem 3 settles existence throughout $k > 1$. The number and location of roots, and existence for $0 < k < 1$, remain distinct questions.

11 Classical candidates and finite-catalog evidence

The direct criterion in Corollary 1 determines the exponents for the three classical candidate center functions without a catalog scan.

Proposition 5. *Among positive exponents, $X(4)$ can satisfy the universal power-detour identity only at $k = 2$, and $X(176)$ only at $k = 1$. The center $X(175)$ cannot satisfy it universally at any $k > 0$.*

Proof. For $X(4)$ use a right triangle with legs b, c, hypotenuse a, and $H = A$. The required identity includes

$$(b^2 + c^2)^{k/2} = b^k + c^k.$$

For positive b, c, strict subadditivity for $0 < k/2 < 1$ and strict superadditivity for $k/2 > 1$ leave only $k = 2$.

For $X(176)$ choose $(a, b, c) = (3, 4, 5)$. The canonical radii are $(3, 2, 1)$ and the inner Soddy completion radius is $6/23$, so the vertex distances are $(75, 52, 29)/23$. A required residual is

$$(75/23)^k + 3^k - (52/23)^k - 4^k.$$

The pairs $(3, 75/23)$ and $(52/23, 4)$ have equal sum, with the latter strictly more spread out. For fixed sum L, the function $t^k + (L - t)^k$ is strictly increasing with spread for $k > 1$, and strictly decreasing with spread for $0 < k < 1$. The residual is therefore nonzero unless $k = 1$.

For $X(175)$ choose $(a, b, c) = (5, 7, 8)$. Here $s = 10$, $\Delta = 10\sqrt{3}$, $R = 7/\sqrt{3}$, and $r = \sqrt{3}$, so $4R + r < 2s$ and the outer Soddy circle encloses the vertex circles. If its radius is R_o, then $PA = R_o - s + a$, cyclically. Since $a < b$, both $PA < PB$ and $a < b$, giving $PA^k + a^k < PB^k + b^k$ for every $k > 0$. □

The familiar pairs $(X(4), 2)$ and $(X(176), 1)$ do satisfy the identity universally by the orthocentric and external-tangency arguments. This proposition settles only the three named center functions, not the rest of the catalog.

11.1 What the supplied catalog records establish

A supplied Wolfram computation used the reference triangle $(6, 9, 13)$ and exponent interval $[1/1000, 100]$ for ETC indices 1 through 72,807. The four production shards have been checked for complete, unique index coverage and against their recorded SHA-256 hashes. A separate 500-entry pilot overlaps the first production shard and is not counted again.

The records report 71,256 interval exclusions, 1,408 nonfinite cases, 86 nonreal cases, 45 missing local definitions, nine unresolved exact evaluations, and three surviving exponent neighborhoods. Separate supplied records report resolutions of the 45 definitions and nine exceptional evaluations, leaving the three classical candidates above.

These are *reported computational outcomes*, not a replayed certificate of the complete finite-catalog theorem. The legacy shards omit exact distance inputs and per-leaf enclosures; some legacy zero and nonreality decisions use non-certifying numerical tests. Complete source-formula data are also external to the supplied archive. Therefore no exhaustive catalog theorem is claimed here. The geometric theorems and the exact classification of the three named candidates are independent of this computation.

For the 45 supplied supplemental formulas, a new computation using Arb real-ball arithmetic generates rational interval subdivisions directly from those formulas and replays every leaf at higher precision. All 45 exclusions on $[1/1000, 100]$ pass this replay. This result applies to the supplied formulas and their documented auxiliary definitions; it does not certify the remaining catalog entries.

The accompanying revised software separates an executable integrity audit from a witness-producing exact/interval workflow. It never promotes legacy status flags to mathematical proof. Its remaining external input is the local formula database and expansion machinery; the required paths, provenance checks, and replay steps are documented in the project README.

12 Questions suggested by the framework

The framework above leaves several natural questions.

1. For a fixed triangle, how does the number of roots of $F_k(\lambda)$ change as k varies? Where do roots merge, appear, or disappear?

2. Is there a distinguished branch P_k varying continuously from the inner Soddy equal-detour point at $k = 1$ to the orthocenter at $k = 2$?

3. For $0 < k < 1$, which power-additive simplices satisfy the closure equation? For $k > 1$, how many admissible roots can occur?

4. Can the power-1 Soddy branch structure and the power-2 Supercenter/orthocentric structure be placed in a common involutive formalism?

5. What is the correct higher-dimensional analogue of the second planar equal-detour point when several positive completion branches exist?

These questions are deliberately separated from the definition. The proved framework is now stronger than a reduction alone: for $k > 1$, existence is equivalent to edge-power additivity. In addition, power-k equal detour implies edge-power additivity; power-additivity reduces the point problem to a one-parameter closure equation; and $k = 1, 2$ recover two established and geometrically rich theories.

Reproducibility and assistance

The accompanying project contains the editable manuscript, figure sources, and verification scripts. Historical computational records are retained with their provenance and limitations. ChatGPT assisted with mathematical development, proof drafting, code revision, and editing. The author must independently review and take responsibility for the final submitted content.

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