The Supercenter:
An In-Excentral Continuation of the Orthocenter Signed Parents, Branches, and Reciprocity

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Abstract

We study the inverse problem of recovering a simplex from prescribed signed centers of spheres tangent to its facet hyperplanes. For a tetrahedron, the resulting Supercenter equations reduce to three opposite-edge pairings in the Gram data of its barycentric gradients. We obtain a complete generic real classification with four projective branches, distinguish finite centers from finite parents, and prove an exact involution between the two quartets of signed centers. The branch through the orthocenter reproduces that point on every nonrectangular orthocentric tetrahedron. Its first differential there agrees with that of the Monge point, although their quadratic separation is generally nonzero. Full reciprocal replacement of each vertex by the selected center characterizes orthocentricity in every dimension. A positive branch exists uniquely for simplices with acute dihedral angles. The planar inverse problem and further orthocentric background are included for comparison. The construction separates sign choices, normalization, and geometric admissibility, making explicit the exceptional loci on which the generic four-branch description or the finite-parent interpretation fails.

Keywords. tetrahedron; orthocentric simplex; barycentric gradient; signed tangent sphere; inverse excentral construction; matrix scaling.
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1 The inverse problem and the main results

For an acute triangle, the orthocenter is the incenter of its orthic parent, whose excenters are the original vertices. For an obtuse triangle the corresponding signed interpretation uses one incenter and two excenters. This inverse construction is rigid in the plane (theorem B.2). In three dimensions it separates from the Euler-line continuation provided by the Monge point. Katsuura's excenter construction [8] supplies the forward geometric relation that we invert here.

The main results are the generic four-branch criterion (theorem 4.1), the signed parent classification (theorem 4.2), the four-branch involution (corollary 5.2), and first-order agreement with Monge on the orthocentric locus (theorem 7.1). Full vertex-replacement reciprocity is characterized in every dimension in theorem 8.1. The explicit radical classification and the two-quartet involution are special to dimension three.

The term *Supercenter* denotes the algebraic construction defined below; a subscript or an explicit branch choice is required whenever a single point is meant. We write $\mathcal{S}_4(T)$ for the entire finite tetrahedral branch set. The names Supercenter and in-ex-parent are project terminology. We make no claim that every equivalent formulation is absent from the literature. All results here have written proofs and do not depend on the separate ETC catalog computation in the companion power-k project.

2 Intrinsic equations and parent reconstruction

Let $T = [Q_0, Q_1, Q_2, Q_3]$ be a nondegenerate tetrahedron, and let λ_i denote its normalized affine barycentric coordinate functions. Set

$$g_i = \nabla \lambda_i, \qquad \sum_i g_i = 0. \tag{1}$$

The vector g_i is an inward normal to the opposite face and has magnitude $1/\eta_i$, where η_i is the corresponding altitude. If ϕ_{ij} is the internal dihedral angle between the faces opposite Q_i, Q_j, define

$$c_{ij} = -g_i \cdot g_j = \frac{\cos \phi_{ij}}{\eta_i \eta_j} \quad (i \neq j). \tag{2}$$

2.1 From a candidate center to a candidate parent

The ordinary forward construction is:

$$(P, I(P)) \longmapsto B(P) = \{E_0, E_1, E_2, E_3\}.$$

We now reverse this construction. Starting only with a tetrahedron

$$T = Q_0 Q_1 Q_2 Q_3,$$

we ask whether there exists a parent tetrahedron P for which the Q_i are the corresponding signed in/excenters, and, if so, whether the distinguished center of the parent can be recovered intrinsically from T. Suppose a point

$$S = \sum_{i=0}^{3} \sigma_i Q_i, \qquad \sum_i \sigma_i = 1, \tag{3}$$

has nonzero barycentric coordinates. If S is to be a signed in/excenter of a parent tetrahedron whose corresponding one-sign-flip centers are Q_i, the parent vertices must be

$$\boxed{A_i' = \frac{S - 2\sigma_i Q_i}{1 - 2\sigma_i}.} \tag{4}$$

This formula is the inverse of Katsuura's excenter formula in the ordinary incenter branch.

To decide whether equation (4) really produces a parent, consider the affine functions

$$F_i(x) = \sum_{k=0}^{3} \frac{\lambda_k(x)}{\sigma_k} - 2\frac{\lambda_i(x)}{\sigma_i}. \tag{5}$$

The plane $F_i = 0$ is the face plane of the candidate parent opposite A_i'. At a reference vertex Q_j,

$$F_i(Q_j) = \begin{cases} -1/\sigma_i, & i = j, \\ 1/\sigma_j, & i \neq j. \end{cases} \tag{6}$$

Thus, for a fixed Q_j, the magnitudes $|F_i(Q_j)|$ are already independent of i. The remaining requirement is that the four face normals have equal lengths, so that the values in equation (6) correspond to equal Euclidean distances.

Write

$$n_i = \nabla F_i = \sum_k \frac{g_k}{\sigma_k} - 2\frac{g_i}{\sigma_i}.$$

A short calculation using equation (2) gives

$$\|n_i\|^2 = C + \frac{4}{\sigma_i}\sum_{j\neq i}\frac{c_{ij}}{\sigma_j}, \tag{7}$$

where C is independent of i.

Definition 2.1 (Tetrahedral Supercenter equations)**.** A projective barycentric vector $[\sigma_0 : \sigma_1 : \sigma_2 : \sigma_3]$ is a *Supercenter branch* of T if

$$\boxed{\frac{1}{\sigma_i}\sum_{j\neq i}\frac{c_{ij}}{\sigma_j} = \rho \qquad (i = 0, 1, 2, 3)} \tag{8}$$

for one common scalar ρ.

Equation equation (7) shows that equation (8) is exactly the equal-normal-length condition. Hence every finite branch with all $\sigma_i \neq 0$ and $1 - 2\sigma_i \neq 0$ reconstructs a finite nondegenerate parent for which the Q_i are signed tangent-sphere centers. A finite center alone does not guarantee a finite parent. Indeed, the matrix of parent barycentrics has determinant $-16\prod_i \sigma_i / \prod_i (1 - 2\sigma_i)$, which is nonzero under these hypotheses.

2.2 Why dimension three is unusually explicit

Put

$$w_{ij} = \frac{c_{ij}}{\sigma_i \sigma_j}.$$

Then equation (8) says that the four weighted degrees of the complete graph K_4 are equal. For K_4 this is equivalent to equality of opposite edge weights:

$$w_{01} = w_{23}, \qquad w_{02} = w_{13}, \qquad w_{03} = w_{12}.$$

Thus the Supercenter equations reduce to

$$c_{01}\sigma_2\sigma_3 = c_{23}\sigma_0\sigma_1, \tag{9a}$$

$$c_{02}\sigma_1\sigma_3 = c_{13}\sigma_0\sigma_2, \tag{9b}$$

$$c_{03}\sigma_1\sigma_2 = c_{12}\sigma_0\sigma_3. \tag{9c}$$

This low-dimensional collapse is the reason a closed radical formula exists for tetrahedra.

Definition 2.2 (Regular tetrahedral domain)**.** For the four-branch results, regularity means that T is nondegenerate, all six c_{ij} are nonzero, the opposite products have compatible signs, and all branch sums and reconstruction denominators used in the statement are nonzero. Projective statements require only nonzero branch coordinates; affine parent statements additionally require $1 - 2\sigma_i \neq 0$. We state explicitly when all four branches must be finite.

3 The eight signed in/excenters of a tetrahedron

3.1 Signed tangent-sphere centers

Let $P = A_0A_1A_2A_3$ be a tetrahedron, and let F_i be the area of the face opposite A_i. For a sign vector

$$\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3), \qquad \varepsilon_i \in \{\pm 1\},$$

define, when the denominator is nonzero,

$$\boxed{C_\varepsilon = \frac{\sum_{i=0}^{3} \varepsilon_i F_i A_i}{\sum_{i=0}^{3} \varepsilon_i F_i}.} \tag{10}$$

The sign vectors ε and $-\varepsilon$ represent the same point, so there are generically $2^3 = 8$ points.

Lemma 3.1 (Equal signed distances)**.** *Every point C_ε in equation (10) is the center of a sphere tangent to the four face planes of P, with the signs ε_i recording on which side of each face plane the center lies.*

Proof. The normalized barycentric coordinate of C_ε at A_i is

$$a_i = \frac{\varepsilon_i F_i}{D_\varepsilon}, \qquad D_\varepsilon = \sum_j \varepsilon_j F_j.$$

The signed distance to the ith face is $a_i h_i$. Since $F_i h_i = 3\,\mathrm{Vol}(P)$,

$$a_i h_i = \varepsilon_i \frac{3\,\mathrm{Vol}(P)}{D_\varepsilon},$$

whose magnitude is independent of i. □

The all-plus point is the ordinary incenter I. The four one-minus points are the four ordinary excenters E_0, E_1, E_2, E_3 in the sense used by Katsuura [8]. The three remaining points may be written

$$X_{01} = X_{23}, \qquad X_{02} = X_{13}, \qquad X_{03} = X_{12},$$

where X_{ij} denotes the sign class with signs i and j reversed.

Because there are four signs, global reversal preserves sign parity. The eight centers therefore split naturally into two tetrahedra:

$$\boxed{A(P) = \{I, X_{01}, X_{02}, X_{03}\}, \qquad B(P) = \{E_0, E_1, E_2, E_3\}.} \tag{11}$$

We shall call these the A- and B-tetrahedra of the parent P.

3.2 The ordinary excentral construction

The facet-area inequality $F_i < \sum_{j \neq i} F_j$ implies that the normalized incenter weights $a_i = F_i / \sum_j F_j$ satisfy $0 < a_i < 1/2$. Katsuura [8, Lemma 3 and Theorem 1] proves that the tetrahedron tangled with a parent about its incenter is its excenter tetrahedron. The affine formula needed here is

$$E_i = \frac{I - 2a_i A_i}{1 - 2a_i}. \tag{12}$$

For normalized barycentrics of any tetrahedron, the *deep interior*, namely the interior of its medial octahedron, is

$$0 < a_i < 1/2 \quad \text{for every } i. \tag{13}$$

The signed formulas use tangent spheres to facet *hyperplanes*; an exterior sphere need not touch every bounded triangular facet.

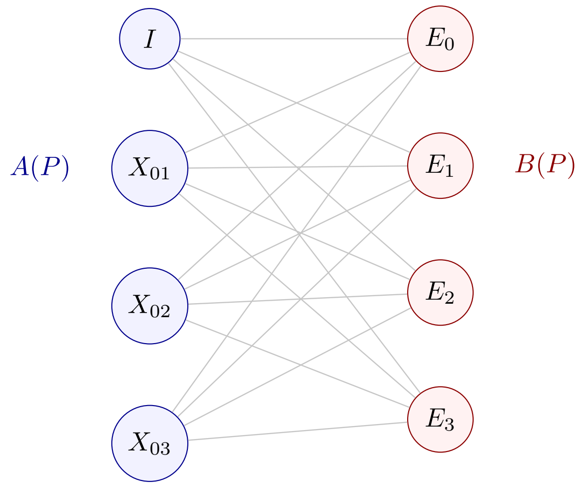


Figure 1: The one-sign-flip graph of the eight signed centers is $K_{4,4}$. Every point of $A(P)$ is adjacent to all four points of $B(P)$ and conversely.

4 Complete generic real classification in dimension three

Throughout this section $c_{ij} \neq 0$. Projective branch counting does not require affine finiteness; statements about finite parents are restricted as in definition 2.2.

4.1 The three opposite-pair signs

Define

$$\boxed{P_1 = c_{01}c_{23}, \qquad P_2 = c_{02}c_{13}, \qquad P_3 = c_{03}c_{12}.} \tag{14}$$

These are products of the dihedral data belonging to opposite pairs.

Theorem 4.1 (Real-branch criterion)**.** *The tetrahedral Supercenter equations have nonzero real projective solutions if and only if*

$$\boxed{\operatorname{sgn} P_1 = \operatorname{sgn} P_2 = \operatorname{sgn} P_3.} \tag{15}$$

When this condition holds there are four real projective branches, counted modulo simultaneous reversal of all signs.

Proof. Taking signs in equation (9) gives

$$\operatorname{sgn}(c_{01}c_{23}) = \operatorname{sgn}(\sigma_0\sigma_1\sigma_2\sigma_3),$$

and the same formula with P_2 and P_3. Hence equation (15) is necessary.

For sufficiency define

$$\boxed{u_i = \sqrt{\left|\prod_{j\neq i} c_{ij}\right|}.} \tag{16}$$

Let p be the common sign in equation (15). Choose signs $\varepsilon_i = \pm 1$ satisfying

$$\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3 = p. \tag{17}$$

Then

$$\boxed{[\sigma_0 : \sigma_1 : \sigma_2 : \sigma_3] = [\varepsilon_0 u_0 : \varepsilon_1 u_1 : \varepsilon_2 u_2 : \varepsilon_3 u_3]} \tag{18}$$

satisfies equation (9). When $L_\varepsilon = \sum_i \varepsilon_i u_i \neq 0$, its normalized weights are $\sigma_i = \varepsilon_i u_i / L_\varepsilon$. Taking absolute values in the pairings determines these four magnitudes up to a common factor, so the list is exhaustive. There are eight sign vectors satisfying equation (17); global reversal gives the same projective point, leaving four branches. □

Geometrically, $P_k > 0$ means that the two dihedral angles in the corresponding opposite pair are either both acute or both obtuse; $P_k < 0$ means that one is acute and the other obtuse. Thus equation (15) is a compatibility condition on the three opposite pairs, not a conventional notion of tetrahedral acuteness.

4.2 The principal branch and the deepness inequalities

Assume first that

$$P_1, P_2, P_3 > 0.$$

The all-plus branch exists and is distinguished:

$$\boxed{\sigma_i^+ = \frac{u_i}{U}, \qquad U = u_0 + u_1 + u_2 + u_3.} \tag{19}$$

It is the unique branch inside T.

Define

$$\boxed{D_i = U - 2u_i = \sum_{j \neq i} u_j - u_i.} \tag{20}$$

At most one D_i can be negative.

Theorem 4.2 (Canonical parent types)**.** *Let T lie in the regular domain of definition 2.2.*

(i) If $P_1, P_2, P_3 > 0$ and $D_i > 0$ for all i, then T is the B-tetrahedron of a unique parent P:

$$T = B(P) = \{E_0, E_1, E_2, E_3\}.$$

Its principal Supercenter is the incenter of the parent and lies in the deep interior of T.

(ii) If $P_1, P_2, P_3 > 0$ and $D_k < 0$ for one k, then T is the A-tetrahedron of a unique parent. The principal branch is still inside T, but lies outside the deep interior; relative to the parent it is an excenter rather than the incenter.

(iii) If $P_1, P_2, P_3 < 0$, then T is again of A-type. All four real Supercenter branches lie outside T.

Proof. For the principal branch set

$$a_i = \frac{1}{2} - \sigma_i^+.$$

Then $\sum a_i = 1$. If all $D_i > 0$, then $0 < \sigma_i^+ < 1/2$, so all $a_i > 0$. In the reconstructed parent equation (4), these are exactly the normalized facet-area weights of the incenter. Consequently the Q_i are the four one-sign-flip excenters: $T = B(P)$. Katsuura's deep-interior criterion equation (13) gives the same inequalities.

If $D_k < 0$, then $\sigma_k^+ > 1/2$, hence $a_k < 0$ while the other three a_i are positive. The principal branch is therefore a one-minus signed center, i.e. an ordinary excenter of the reconstructed parent.

Table 1: Generic tetrahedral Supercenter regimes.

Region	Canonical role of T	Real Supercenter geometry
$\mathscr{B}$	$B = \{E_0, E_1, E_2, E_3\}$	one deep-interior branch I, three exterior
$\mathscr{A}_k^+$	$A = \{I, X_{01}, X_{02}, X_{03}\}$	one shallow interior branch, three exterior
$\mathscr{A}^-$	A-type	four exterior branches
$\mathscr{N}$	no real in-ex-parent	no real branches

Flipping one sign at a time changes parity, so the four reference vertices belong to the opposite parity class $A(P)$.

If all $P_i < 0$, equation (17) forces every branch to have mixed signs. The same one-sign-flip interpretation gives A-type. Uniqueness follows because any one finite branch reconstructs the parent face planes; the other three branches are the remaining signed centers in the opposite parity class and reconstruct the same four parent vertices up to relabeling. □

Definition 4.3 (Phase-space terminology)**.** For generic tetrahedra we use:

$$\begin{aligned}
\mathscr{B} &= \{P_i > 0 \text{ for all } i,\ D_j > 0 \text{ for all } j\},\\
\mathscr{A}_k^+ &= \{P_i > 0 \text{ for all } i,\ D_k < 0\},\\
\mathscr{A}^- &= \{P_i < 0 \text{ for all } i\},\\
\mathscr{N} &= \{\operatorname{sgn} P_1, \operatorname{sgn} P_2, \operatorname{sgn} P_3 \text{ not all equal}\}.
\end{aligned}$$

We call these respectively the B-domain, shallow A-lobes, alternating A-domain, and non-real domain.

4.3 Where is the Supercenter?

The location is immediate from barycentric signs.

Corollary 4.4 (Interior/exterior classification)**.** *Away from singular boundaries:*

(i) If $P_1, P_2, P_3 > 0$, exactly one Supercenter branch lies inside T; it is the principal branch equation (19).

(ii) If $P_1, P_2, P_3 < 0$, all four real branches lie outside T.

(iii) If the signs are incompatible, no real branch exists.

Moreover the interior branch is deep precisely when $D_i > 0$ for all i.

4.4 Boundary hypersurfaces

Three types of boundaries deserve separate names.

1. **Right-dihedral walls:**

$$c_{ij} = 0.$$

A dihedral angle is 90°, and the equations lose rank. These are the direct three-dimensional analogues of the right-triangle boundary.

2. A/B **transition walls:**
$$D_i = 0 \iff u_i = \sum_{j \neq i} u_j.$$
Then $\sigma_i^+ = 1/2$ and the corresponding denominator in equation (4) vanishes. The parent becomes degenerate in that direction.

3. **Branches at infinity:** for a signed branch,
$$L_\varepsilon = \sum_i \varepsilon_i u_i = 0.$$
The normalized barycentrics cannot be made to sum to 1, so the branch is a projective point at infinity. A genuine parent can still exist; geometrically this corresponds to a signed facet-area sum vanishing. Katsuura's "weakly reversible" condition is closely related to such equal-sum phenomena [8].

5 A second three-dimensional surprise: exact eight-point A/B duality

Retaining all four branches produces a finite eight-point configuration on each regular domain. The two signed-center quartets are exchanged by the Supercenter operation.

For a fixed parent P, equation (11) gives two canonical tetrahedra. The one-sign-flip graph in figure 1 explains the strongest three-dimensional analogue of the planar in/excentral quadrangle.

Proposition 5.1 (A/B Supercenter duality)**.** *Assume all relevant signed centers are finite and both $A(P), B(P)$ are nondegenerate with all their off-diagonal c_{ij} nonzero. Then the four algebraic Supercenter branches of $B(P)$ are exactly the four vertices of $A(P)$, and the four branches of $A(P)$ are exactly the four vertices of $B(P)$.*

Proof. Choose any signed center C_ε of the parent and let $C_{\varepsilon^{(i)}}$ denote the center obtained by flipping the ith sign. With
$$a_i = \frac{\varepsilon_i F_i}{\sum_j \varepsilon_j F_j}, \qquad \sum_i a_i = 1,$$
one has the exact relation
$$C_{\varepsilon^{(i)}} = \frac{C_\varepsilon - 2a_i A_i}{1 - 2a_i}. \tag{21}$$
Thus C_ε is a Supercenter branch of the tetrahedron formed by its four one-sign-flip neighbors. In dimension three the antipodal quotient of the four-cube is $K_{4,4}$, so the neighbors of every point of $A(P)$ are all four points of $B(P)$ and conversely. These four distinct solutions exhaust the branches by theorem 4.1. $\square$

It is useful to package the four branches into a single operation. For a regular real-parentable tetrahedron T, let
$$\mathcal{S}_4(T)$$
denote the unordered tetrahedron formed by its four finite real Supercenter branches.

Corollary 5.2 (Exact four-branch involution)**.** *On every chamber on which the parent and all four branches are finite and nondegenerate,*

$$\boxed{\mathcal{S}_4(\mathcal{S}_4(T)) = T} \tag{22}$$

as an unordered tetrahedron. In particular,

$$\mathcal{S}_4(A(P)) = B(P), \qquad \mathcal{S}_4(B(P)) = A(P).$$

5.1 Hadamard form of the four branches

There is a compact affine formula for this involution. Choose one finite branch

$$S = \sum_{i=0}^{3} \sigma_i Q_i, \qquad \sum_i \sigma_i = 1,$$

with all $\sigma_i \neq 0$. Relative to this branch, the four allowed sign classes differ by an even number of sign reversals. Introduce

$$\mathsf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \qquad \mathsf{H}_4\mathsf{H}_4^T = 4I. \tag{23}$$

Let h_{si} be its entries and put

$$z_s = \sum_{i=0}^{3} h_{si}\sigma_i. \tag{24}$$

Whenever $z_s \neq 0$, the four branches are

$$\boxed{R_s = \frac{\sum_i h_{si}\sigma_i Q_i}{z_s}, \qquad s = 0, 1, 2, 3.} \tag{25}$$

Here $R_0 = S$. If Q and R denote the column vectors of the four vertices and branches, respectively, then

$$R = Z^{-1}\mathsf{H}_4\Sigma Q, \qquad \Sigma = \operatorname{diag}(\sigma_0, \dots, \sigma_3), \quad Z = \operatorname{diag}(z_0, \dots, z_3). \tag{26}$$

Consequently

$$Q = \frac{1}{4}\Sigma^{-1}\mathsf{H}_4^T Z R. \tag{27}$$

Thus the barycentric coordinates of Q_i with respect to the branch tetrahedron $R_0R_1R_2R_3$ are

$$\boxed{\gamma_s^{(i)} = \frac{h_{si}z_s}{4\sigma_i}.} \tag{28}$$

The matrix identity itself is affine. The fact that the Q_i are not merely affine inverse images but are the four *Supercenter branches* of $R_0R_1R_2R_3$ follows from proposition 5.1, i.e. from the common signed-center parent and the one-sign-flip relation equation (21).

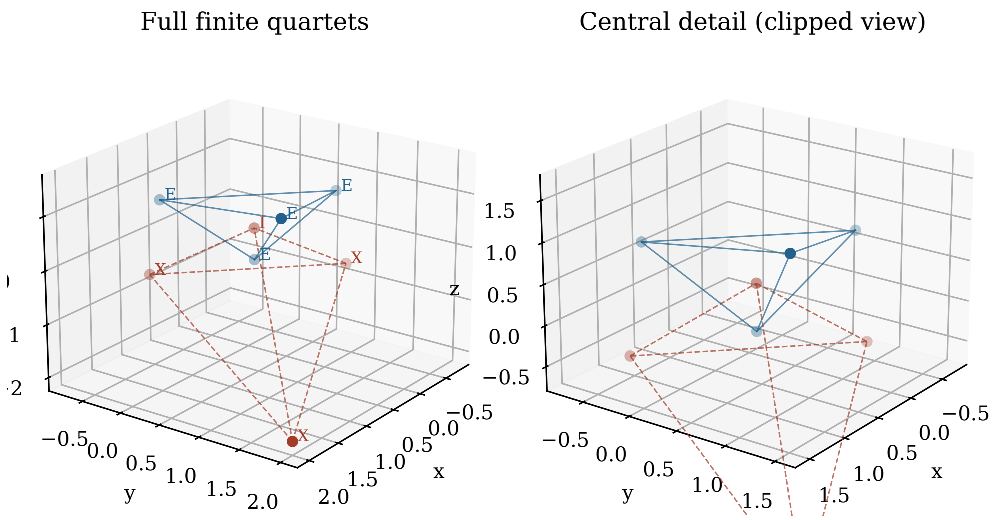


Figure 2: Computed signed quartets for the rational parent in example 6.3. The right panel enlarges the central region and clips the more distant points. The two quartets are individually finite and affinely independent; dashed edges indicate the complementary signed quartet, not sphere tangencies.

6 The Supercenter on the orthocentric locus

A tetrahedron is orthocentric if its four altitudes concur. Equivalently, its three pairs of opposite edges are perpendicular, or there are vertex potentials α_i with $|Q_i - Q_j|^2 = \alpha_i + \alpha_j$; the proof is recorded in proposition C.1. With the orthocenter as origin, all off-diagonal vertex scalar products are equal. A *rectangular* orthocentric simplex has its orthocenter at a vertex and is excluded from the nonzero-weight formulas.

The Monge point of a tetrahedron is

$$M = 2G - O, \tag{29}$$

where G and O are the centroid and circumcenter. It coincides with the orthocenter on the orthocentric locus [2, 4].

Theorem 6.1 (Orthocentric reproduction)**.** *Let $T = Q_0Q_1Q_2Q_3$ be a nonrectangular orthocentric tetrahedron with orthocenter*

$$H = \sum_{i=0}^{3} h_i Q_i, \qquad \sum_i h_i = 1.$$

Then H is a Supercenter branch of T. More precisely, there is a nonzero constant κ such that

$$\boxed{c_{ij} = \kappa h_i h_j \qquad (i \neq j).} \tag{30}$$

Consequently

$$P_1 = P_2 = P_3 = \kappa^2 h_0 h_1 h_2 h_3.$$

Proof. Put the orthocenter at the origin and write $v_i = Q_i - H$. By proposition C.1, $v_i \cdot v_j = c$ for $i \neq j$. Since $\sum h_i v_i = 0$, evaluation of the barycentric coordinate functions at the vertices gives

$$g_i = -\frac{h_i}{c} v_i.$$

Thus, for $i \neq j$,

$$c_{ij} = -g_i \cdot g_j = -\frac{h_i h_j}{c},$$

which is equation (30) with $\kappa = -1/c$. Substitution into equation (8) gives the same value of ρ for every i, so H is a Supercenter branch. $\square$

Corollary 6.2 (When an orthocentric tetrahedron is a four-excenter tetrahedron)**.** *Suppose the orthocenter is interior, so $h_i > 0$. Then the following are equivalent:*

(i) T belongs to the B-domain and is the four-excenter tetrahedron of a genuine parent;

(ii) $0 < h_i < 1/2$ for all i;

(iii) the circumcenter of T is interior.

In this case the parent satisfies

$$\boxed{I(\textit{parent}) = S(T) = H(T).}$$

Proof. By equation (30), the quantities u_i are proportional to h_i, so the deep inequalities $D_i > 0$ are exactly $h_i < 1/2$. Edmonds–Hajja–Martini prove that for an orthocentric d-simplex the circumcenter is interior precisely when the orthocenter barycentrics satisfy $0 < h_i < 1/(d-1)$ [4, Theorem 3.10]; for $d = 3$ this is $h_i < 1/2$. $\square$

This gives a precise meaning to the informal phrase "adequately acute orthocentric tetrahedron": it is exactly a well-centered orthocentric tetrahedron, i.e. one whose circumcenter lies inside.

6.1 The Supercenter is not the Monge point

Both M and S reproduce H on the orthocentric locus, but they are different continuations.

Example 6.3 (A rational nonorthocentric B-tetrahedron)**.** Start with the parent

$$A' = (0,0,0), \quad B' = (1,0,0), \quad C' = (0,1,0), \quad D' = (0,0,12).$$

Its opposite face areas have ratio

$$17 : 12 : 12 : 1,$$

so

$$I' = \left(\frac{2}{7}, \frac{2}{7}, \frac{2}{7}\right).$$

Its four excenters are

$$Q_0 = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right), \qquad Q_1 = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right),$$

$$Q_2 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right), \qquad Q_3 = \left(\frac{3}{10}, \frac{3}{10}, -\frac{3}{10}\right).$$

The reference tetrahedron $T = Q_0Q_1Q_2Q_3$ is not orthocentric, since

$$(Q_1 - Q_0) \cdot (Q_3 - Q_2) = \frac{143}{180} \neq 0.$$

Its intrinsic data are

$$\begin{aligned} c_{01} &= 3/34, & c_{02} &= 3/34, & c_{03} &= 5/306, \\ c_{12} &= 9/544, & c_{13} &= 15/34, & c_{23} &= 15/34. \end{aligned}$$

Hence

$$u_0 : u_1 : u_2 : u_3 = 4 : 9 : 9 : 20$$

and

$$S = [4 : 9 : 9 : 20] = \left(\frac{2}{7}, \frac{2}{7}, \frac{2}{7}\right) = I'.$$

The centroid, circumcenter and Monge point of T are

$$G = \left(\frac{9}{20}, \frac{9}{20}, \frac{19}{30}\right),$$

$$O = \left(\frac{87}{136}, \frac{87}{136}, \frac{161}{170}\right),$$

and

$$M = 2G - O = \left(\frac{177}{680}, \frac{177}{680}, \frac{163}{510}\right).$$

Thus $S \neq M$. Finally, substituting the normalized weights

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = \left(\frac{2}{21}, \frac{3}{14}, \frac{3}{14}, \frac{10}{21}\right)$$

into equation (4) reconstructs exactly the original parent $A'B'C'D'$.

7 Infinitesimal deformation away from orthocentricity

The preceding sections show that S and M agree on the orthocentric locus but need not agree away from it. A natural local question is therefore: *at what order do the two continuations separate?* The answer is unexpectedly strong. At every regular orthocentric tetrahedron they have the same first differential.

7.1 A two-parameter normal form

Let an orthocentric tetrahedron be placed, by a Euclidean similarity, in the form

$$A = (-1, 0, 0), \qquad B = (1, 0, 0), \qquad C = (x, y, 0), \qquad D_0 = (x, t, h), \tag{31}$$

where $y > 0$, $h > 0$, and

$$t = \frac{1 - x^2}{y}, \qquad r = x^2 + y^2 - 1, \qquad q = t(y - t) = \frac{(1 - x^2)r}{y^2}. \tag{32}$$

Indeed $(x, t, 0)$ is the orthocenter of the face ABC, so the orthogonal projection of D_0 to the base plane must be $(x, t, 0)$. The tetrahedral orthocenter is

$$H_0 = \left(x, t, \frac{q}{h}\right). \tag{33}$$

Now perturb only the horizontal projection of D:

$$D(e,f) = (x+e, t+f, h). \tag{34}$$

The two independent opposite-edge orthogonality defects may be chosen as

$$\Phi_1 = (B-A)\cdot(D-C) = 2e, \tag{35}$$
$$\Phi_2 = (C-A)\cdot(D-B) = (1+x)e + yf. \tag{36}$$

The third opposite-edge condition follows from the identity

$$(AB)\cdot(CD) - (AC)\cdot(BD) + (AD)\cdot(BC) = 0.$$

The Jacobian of (Φ_1, Φ_2) with respect to (e,f) is

$$\begin{pmatrix} 2 & 0 \\ 1+x & y \end{pmatrix}, \qquad \det = 2y \neq 0.$$

Thus (e,f) are genuine local coordinates normal to the orthocentric locus. A vertical displacement of D changes h but stays within the orthocentric family.

For this perturbation the Monge point can be calculated exactly:

$$\boxed{M(e,f) = \left(x + \frac{e}{2}, t + \frac{f}{2}, \frac{q}{h} - \frac{x}{h}e + \frac{3x^2+y^2-3}{2hy}f - \frac{e^2+f^2}{2h}\right).} \tag{37}$$

On the analytic Supercenter branch through H_0, expansion of the branch formula equation (18) gives

$$\boxed{S(e,f) = H_0 + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ -\frac{x}{h} & \frac{3x^2+y^2-3}{2hy} \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} + O\Big(\|(e,f)\|^2\Big).} \tag{38}$$

Comparing equations (37) and (38) already proves equality of the two normal derivatives. For completeness, the differentiation rule giving this expansion is

$$L_i = \frac{1}{2}\sum_{j\neq i} \frac{\partial c_{ij}}{c_{ij}}, \qquad \partial\sigma_i = h_i\left(L_i - \sum_j h_j L_j\right), \qquad \partial S = \sum_i (\partial\sigma_i) Q_i + \sum_i h_i \partial Q_i.$$

Here h_i are the orthocenter weights and the derivatives are evaluated at the undeformed tetrahedron. This follows by logarithmically differentiating the signed radical formula; the sign chamber is fixed. Substitution of the barycentric gradients of equation (31) gives equation (38) by rational simplification. The accompanying exact verification script checks both normal directions symbolically in x, y, h.

7.2 First-order tangency theorem

Theorem 7.1 (First-order tangency of Supercenter and Monge point)**.** *Let T_0 be a nonrectangular orthocentric tetrahedron at which the chosen real Supercenter branch is finite and analytic, and require $S(T_0) = H(T_0)$. For every differentiable deformation $T(\varepsilon)$ with $T(0) = T_0$,*

$$\boxed{S(T(\varepsilon)) - M(T(\varepsilon)) = O(\varepsilon^2).} \tag{39}$$

Equivalently,

$$DS_{T_0} = DM_{T_0}.$$

Thus S and M have the same first jet along the regular orthocentric locus.

Proof. Let $F(T) = S(T) - M(T)$. The orthocentric locus is locally of codimension two: in the normalization equation (31), equations (35) and (36) have independent differentials. Since F vanishes identically on the orthocentric locus by theorem 6.1, DF_{T_0} vanishes on every tangent deformation of that locus. It remains to test a basis for the two-dimensional normal quotient. The e- and f-directions in equation (34) form such a basis, and equations (37) and (38) show that DF_{T_0} vanishes in both directions. Hence $DF_{T_0} = 0$ on the full deformation space. Ambient analyticity now gives $F(T) = O(\|T - T_0\|^2)$. Since a differentiable deformation satisfies $\|T(\varepsilon) - T_0\| = O(|\varepsilon|)$, the stated quadratic remainder follows. $\square$

The theorem corrects a tempting but inaccurate first impression: although a generic nonorthocentric tetrahedron has $S \neq M$, the difference does *not* appear linearly when one leaves orthocentricity. The first possible separation is quadratic. It is generically nonzero. For example, taking

$$x = \frac{1}{5}, \qquad y = 1, \qquad h = \frac{6}{5}, \qquad f = 0,$$

a direct series calculation gives

$$S - M = \left(\frac{8875}{7008}, \frac{1675}{584}, -\frac{211105}{21024}\right) e^2 + O(e^3). \tag{40}$$

Thus the second normal derivative contains genuine information not present in the Monge construction.

7.3 The four reciprocal tetrahedra

Let $T(\varepsilon) = Q_0(\varepsilon)Q_1(\varepsilon)Q_2(\varepsilon)Q_3(\varepsilon)$ deform an orthocentric tetrahedron, and put $S(\varepsilon) = S(T(\varepsilon))$. For each i, form

$$T_i(\varepsilon) = [S(\varepsilon), Q_0(\varepsilon), \ldots, \widehat{Q_i(\varepsilon)}, \ldots, Q_3(\varepsilon)].$$

At $\varepsilon = 0$ the five points form an orthocentric system, so $T_i(0)$ is orthocentric and has orthocenter $Q_i(0)$. Applying theorem 7.1 to the deformation $T_i(\varepsilon)$ and using the chain rule yields

$$\boxed{S(T_i(\varepsilon)) - M(T_i(\varepsilon)) = O(\varepsilon^2) \qquad (i = 0, 1, 2, 3).} \tag{41}$$

The five-point reciprocal property itself is much less stable: it is destroyed at first order in a transverse deformation. In the normal form above, for the tetrahedron $T_D = SABC$ one finds

$$S(T_D) - D = M(T_D) - D + O(\|(e,f)\|^2) = K_D \begin{pmatrix} e \\ f \end{pmatrix} + O(\|(e,f)\|^2), \tag{42}$$

where

$$\boxed{K_D = \begin{pmatrix} -3/4 & 0 \\ 0 & -3/4 \\ \dfrac{hx}{2q} & -\dfrac{h(3x^2 + y^2 - 3)}{4qy} \end{pmatrix}.} \tag{43}$$

The first two rows already show that K_D has rank two. Thus no nonzero transverse infinitesimal displacement preserves even this one reciprocal relation. After relabeling and renormalizing the coordinates the same conclusion holds at each of the other three positions. This local calculation is consistent with the global rigidity theorem in the next section.

8 Rigidity of full vertex-replacement reciprocity

A configuration $\{Q_0, \dots, Q_d, S\}$ is *Super-reciprocal* if S is a Supercenter of $[Q_0, \dots, Q_d]$ and each omitted Q_i is a Supercenter after replacing Q_i by S. All constituent simplices and barycentric coordinates used below are nondegenerate and nonzero, respectively.

Theorem 8.1 (Full Supercenter reciprocity). *Let $T = [Q_0, \dots, Q_d]$ and let $S = \sum_i \sigma_i Q_i$, where $\sum_i \sigma_i = 1$ and every $\sigma_i \neq 0$. Use the algebraic Supercenter definition*

$$\sum_{j \neq i} \frac{c_{ij}}{\sigma_i \sigma_j} = \rho, \qquad c_{ij} = -g_i \cdot g_j, \quad g_i = \nabla \lambda_i.$$

Then S is a Supercenter of T and every Q_i is a Supercenter of the simplex obtained by replacing Q_i by S if and only if T is orthocentric with orthocenter S. The nonzero-coordinate assumptions exclude the degenerate orthocentric systems.

Proof. For necessity assume all reciprocal equations hold. Put $w_{ij} = c_{ij}/(\sigma_i \sigma_j)$, so each original weighted degree is ρ. On replacing Q_0 by S, the new gradients and the barycentrics of Q_0 are

$$g_0' = g_0/\sigma_0, \quad g_i' = g_i - (\sigma_i/\sigma_0) g_0, \quad \tau_0 = 1/\sigma_0, \quad \tau_i = -\sigma_i/\sigma_0.$$

Let $b = \|g_0\|^2$. Direct calculation gives

$$w_{0i}' = -\sigma_0^2 w_{0i} - b, \quad w_{ij}' = \sigma_0^2 (w_{ij} - w_{0i} - w_{0j}) - b \quad (i, j > 0).$$

The new weighted degrees are

$$r_0' = -\sigma_0^2 \rho - db, \qquad r_i' = -d\sigma_0^2 w_{0i} - db \quad (i > 0).$$

Thus reciprocity at position 0 implies $w_{0i} = \rho/d$. Repeating for each position implies

$$c_{ij} = \kappa \sigma_i \sigma_j, \qquad \kappa = \rho/d.$$

Because $\sum_i g_i = 0$, this also gives $\|g_i\|^2 = \kappa \sigma_i (1 - \sigma_i)$. If $\kappa = 0$ all gradients would vanish, which is impossible. Both vectors $Q_i - S$ and $g_i/(\kappa \sigma_i)$ have scalar products $\delta_{ij} - \sigma_j$ with each g_j. Since the gradients span the ambient space, these vectors coincide. Hence

$$(Q_i - S) \cdot (Q_j - S) = -1/\kappa \quad (i \neq j),$$

which implies altitude concurrence at S.

Conversely, a nondegenerate orthocentric system remains orthocentric under each replacement, with the omitted point as orthocenter. The orthocentric factorization $c_{ij} = \kappa \sigma_i \sigma_j$ shows directly that its orthocenter satisfies the balancing equations. Apply this to each constituent simplex. □

Remark 8.2. For $d = 3$, one reciprocal replacement already suffices, because the original K_4 balance equations identify opposite edge weights. In higher dimension the proof above uses reciprocity at every position. It does not classify all signed branches.

Thus in three dimensions the selected five-point property is rigid, although the full four-branch operation is involutive on an open regular domain.

9 Preliminary results in higher dimensions

We now give the general parent interpretation and positive-branch existence theorem. The full reciprocity characterization was proved in theorem 8.1. Throughout $d \geq 2$.

9.1 Signed centers and the higher-dimensional neighbor graph

Let

$$P = [A_0, \ldots, A_d] \subset \mathbb{R}^d$$

be a nondegenerate d-simplex, and let F_i be the $(d-1)$-dimensional volume of the facet opposite A_i. For a sign vector $\varepsilon \in \{\pm 1\}^{d+1}$ define

$$C_\varepsilon = \frac{\sum_{i=0}^{d} \varepsilon_i F_i A_i}{\sum_{i=0}^{d} \varepsilon_i F_i}, \tag{44}$$

when finite. Again $C_\varepsilon = C_{-\varepsilon}$, so there are generically

$$\boxed{2^d}$$

signed tangent-hypersphere centers. The proof of lemma 3.1 is unchanged because

$$F_i h_i = d \ \mathrm{Vol}_d(P).$$

Connect two sign classes when one sign flip relates them. The resulting graph is the antipodal quotient of the $(d+1)$-dimensional hypercube. Every center has exactly $d+1$ neighbors.

d	signed centers	one-sign-flip structure
2	4	K_4: every center sees the other three
3	8	$K_{4,4}$: two complementary tetrahedra
4	16	degree 5 quotient graph; no two-simplex collapse
d	2^d	degree $d+1$ antipodal cube quotient

Thus Guy's quadrangle is exceptional: in dimension two the neighbor set of a signed center is literally the set of all other signed centers. In dimension three it becomes the $A \leftrightarrow B$ pair. For $d \geq 4$ the symmetry remains local but no longer collapses to one quadrangle or one pair of complementary simplices.

9.2 The dimension-independent Supercenter equation

Let

$$T = [Q_0, \ldots, Q_d]$$

be a reference simplex with barycentric gradients $g_i = \nabla \lambda_i$ and

$$c_{ij} = -g_i \cdot g_j \qquad (i \neq j).$$

For normalized nonzero weights σ_i, set $S = \sum_i \sigma_i Q_i$ and $A_i' = (S - (d-1)\sigma_i Q_i)/(1-(d-1)\sigma_i)$. The candidate facet functions are

$$F_i = \sum_j \lambda_j / \sigma_j - 2\lambda_i / \sigma_i.$$

Indeed $F_i(A'_j) = 0$ when $i \neq j$, and $F_i(A'_i) = 2(d-1)/(1-(d-1)\sigma_i)$. Their normal vectors retain the same coefficient 2 in every dimension. Equality of their lengths gives precisely the following equation. A finite parent requires every $1-(d-1)\sigma_i \neq 0$.

Definition 9.1 (d-dimensional Supercenter)**.** A barycentric vector $[\sigma_0 : \cdots : \sigma_d]$ is a Supercenter branch if

$$\boxed{\frac{1}{\sigma_i}\sum_{j\neq i}\frac{c_{ij}}{\sigma_j} = \rho \qquad (i = 0, \ldots, d)} \tag{45}$$

for one scalar ρ.

Putting $z_i = 1/\sigma_i$, this becomes

$$\boxed{z_i(Cz)_i = \rho,} \tag{46}$$

where C is the symmetric matrix with zero diagonal and off-diagonal entries c_{ij}. Equivalently, the edge weights

$$w_{ij} = c_{ij}z_iz_j$$

on the complete graph K_{d+1} must have equal weighted degree at every vertex. This graph-balancing formulation appears to be the most natural dimension-independent expression of the construction.

9.3 Reconstruction and the generalized deep condition

Suppose $S = \sum \sigma_i Q_i$ is a branch associated with an ordinary parent in which S is the incenter. If a_i are the normalized facet hyperareas of the parent, then the one-sign-flip centers satisfy

$$Q_i = \frac{S - 2a_iA'_i}{1-2a_i}.$$

Since $\sum a_i = 1$,

$$\sum_{i=0}^{d}(1-2a_i) = d-1.$$

Therefore

$$\boxed{\sigma_i = \frac{1-2a_i}{d-1}} \tag{47}$$

and

$$\boxed{A'_i = \frac{S-(d-1)\sigma_iQ_i}{1-(d-1)\sigma_i}.} \tag{48}$$

For an ordinary parent the facet-area inequality $F_i < \sum_{j\neq i} F_j$ gives $0 < a_i < 1/2$, hence

$$\boxed{0 < \sigma_i < \frac{1}{d-1}.} \tag{49}$$

For $d = 2$ the upper bound is automatic; for $d = 3$ it is precisely Katsuura's deep-interior condition; for higher d it defines the natural excentral-admissibility polytope in barycentric space.

9.4 Orthocentric simplices are reproduced in every dimension

Theorem 9.2 (Higher-dimensional orthocentric reproduction)**.** *Let $T = [Q_0, \dots, Q_d]$ be a nonrectangular orthocentric d-simplex with orthocenter*

$$H = \sum_{i=0}^{d} h_i Q_i.$$

Then H satisfies the Supercenter equation equation (45).

Proof. Put $H = 0$. The orthocentric vector criterion gives $Q_i \cdot Q_j = c$ for $i \neq j$. Exactly as in the proof of theorem 6.1,

$$g_i = -\frac{h_i}{c} Q_i, \qquad c_{ij} = \kappa h_i h_j$$

for $\kappa = -1/c$. Substituting $\sigma_i = h_i$ into equation (45) gives

$$\frac{1}{h_i} \sum_{j \neq i} \frac{\kappa h_i h_j}{h_j} = d\kappa,$$

independent of i. □

Corollary 9.3 (Well-centered orthocentric simplices)**.** *Let an orthocentric d-simplex have interior orthocenter. It is the ordinary $(d+1)$-excenter simplex of a parent, with parent incenter equal to its orthocenter, precisely when*

$$0 < h_i < \frac{1}{d-1} \qquad \textit{for every } i.$$

By Edmonds–Hajja–Martini [4, Theorem 3.10], this is equivalent to the circumcenter of the orthocentric simplex being interior.

This single statement recovers two important low-dimensional facts:

- for $d = 2$, an orthocentric simplex is just a triangle and the condition reduces to acuteness, so every acute triangle is EEE;
- for $d = 3$, it becomes the deep condition $h_i < 1/2$ found in corollary 6.2.

9.5 Existence and uniqueness on the positive acute branch

When all $c_{ij} > 0$, the positive higher-dimensional branch has a useful variational formulation. Put $z_i = e^{y_i}$ and consider

$$\Phi(y) = \sum_{i<j} c_{ij} e^{y_i + y_j} \tag{50}$$

subject to

$$\sum_i y_i = 0.$$

The Lagrange equations are precisely

$$z_i \sum_{j \neq i} c_{ij} z_j = \rho,$$

i.e. equation (46). Moreover

$$v^T (\nabla^2 \Phi) v = \sum_{i<j} c_{ij} e^{y_i + y_j} (v_i + v_j)^2 > 0$$

for nonzero v when $d+1 \geq 3$. Hence Φ is strictly convex. It is also coercive on $\sum y_i = 0$: every nonzero vector in this hyperplane has a pair with positive sum. Otherwise, taking its largest coordinate $m > 0$, every other coordinate would be at most $-m$, contradicting zero total for at least three coordinates. Compactness of the unit sphere in the hyperplane gives a uniform positive lower bound for the maximum pair sum. Thus $\Phi(y) \to \infty$ as $\|y\| \to \infty$, and a unique positive solution exists. Standard implicit-function arguments then show that this positive Supercenter varies real-analytically with the metric data throughout the region $c_{ij} > 0$.

This is a symmetric diagonal scaling problem: with $D = \mathrm{diag}(z)$, the matrix DCD has constant row sums and, after scalar normalization, is doubly stochastic. The general theory of nonnegative matrix scaling is classical [10]; the convex argument above is self-contained and handles the zero diagonal directly.

Proposition 9.4 (Positive-branch existence in all dimensions)**.** *Every simplex with all internal dihedral angles acute has a unique positive Supercenter branch. This branch need not satisfy the stronger parent condition equation (49).*

9.6 The signed-center neighbor theorem

The most symmetric higher-dimensional statement is local on the 2^d signed centers.

Theorem 9.5 (Neighbor-simplex property)**.** *Let $P = [A_0, \dots, A_d]$ be a parent simplex and let C_ε be one finite signed center. Let $\varepsilon^{(i)}$ denote the sign vector obtained by flipping the ith sign. Whenever the $d+1$ points $C_{\varepsilon^{(i)}}$ are affinely independent, the center C_ε is a Supercenter branch of the neighbor simplex*

$$[C_{\varepsilon^{(0)}}, \dots, C_{\varepsilon^{(d)}}].$$

Proof. Set

$$a_i = \frac{\varepsilon_i F_i}{\sum_j \varepsilon_j F_j}.$$

Directly from equation (44),

$$C_{\varepsilon^{(i)}} = \frac{C_\varepsilon - 2a_i A_i}{1 - 2a_i}.$$

This is exactly the inverse-center relation underlying equation (45); hence C_ε satisfies the Supercenter equations of its neighbor simplex. □

For $d = 2$ the neighbor simplex of each center consists of the other three centers, giving Guy's full four-point symmetry. For $d = 3$ the neighbor graph is $K_{4,4}$ and theorem 9.5 becomes the $A \leftrightarrow B$ duality of proposition 5.1. For $d \geq 4$ this theorem predicts a richer network of overlapping neighbor simplices rather than a single quadrangle or a single dual pair.

9.7 What is still open in higher dimension?

The higher-dimensional equation equation (45) is simple to state, but the complete real classification is no longer as elementary as for tetrahedra. In dimension three equal weighted degrees on K_4 force opposite edge weights to match, yielding the explicit product formula equation (18). For K_{d+1} with $d \geq 4$, equal weighted degrees leave many more degrees of freedom and there is no analogous reduction to three pairings.

Three natural problems remain:

1. classify all real signed branches of equation (45) and the corresponding parent types;

2. characterize the singular hypersurfaces on which branches merge, become infinite, or the inverse parent degenerates;
3. determine the geometric and combinatorial structure of overlapping neighbor simplices in dimensions $d \geq 4$, beyond the full reciprocity characterization in theorem 8.1.

A tempting but generally incorrect shortcut is

$$\widetilde{\sigma}_i \propto \left(\prod_{j\neq i} c_{ij}\right)^{1/(d-1)}.$$

On the positive branch this formula reproduces the orthocenter on the interior orthocentric locus and solves the low-dimensional balancing equations for $d = 2, 3$, but for $d \geq 4$ it does not in general solve the balancing equation equation (45). The matrix-balancing formulation, not the product formula, is therefore the natural extrapolation.

10 Discussion

The inverse signed-center relation admits two different reciprocity statements. Full vertex replacement is rigid and characterizes orthocentric systems in every dimension. In three dimensions, retaining all four projective branches instead gives a two-quartet involution on the finite regular domain. The branch through H agrees with Monge to first order along orthocentricity, so its second normal derivative measures the first local separation of the two continuations.

The main unresolved questions concern nonregular boundaries, global classification of signed branches in higher dimension, and their geometric locations. The companion power-k framework studies a different construction, based on $d_{ij}^k = u_i + u_j$; its $k = 2$ carrier is the same orthocentric family, but that common specialization does not by itself identify the two constructions away from orthocentricity.

A Planar preliminaries

A.1 Barycentric coordinates and center functions

Let $\Delta = ABC$ be a nondegenerate triangle. Every point P in its affine plane has normalized barycentric coordinates

$$P = x_A A + x_B B + x_C C, \qquad x_A + x_B + x_C = 1.$$

We write the corresponding homogeneous coordinates as

$$P = [x_A : x_B : x_C].$$

The point is interior precisely when the three normalized coordinates are positive.

Many triangle centers are given by a homogeneous center function. If the side lengths opposite A, B, C are a, b, c, a function $f(a; b, c)$ gives the barycentrics

$$[f(a; b, c) : f(b; c, a) : f(c; a, b)].$$

The centroid is simply

$$G = [1 : 1 : 1].$$

For a nonright triangle the orthocenter can be written

$$H = [\tan A : \tan B : \tan C]. \tag{51}$$

Equivalently, since $\tan A = 4\Delta/(b^2 + c^2 - a^2)$,

$$f_H(a; b, c) = \frac{1}{b^2 + c^2 - a^2}$$

may be used as a homogeneous center function.

A.2 Recovering the orthocenter from the Euler line

For a triangle, G and O determine H by

$$H = 3G - 2O. \tag{52}$$

The circumcenter has barycentrics

$$O = [\sin 2A : \sin 2B : \sin 2C].$$

Let its normalized coordinates be $O = [o_A, o_B, o_C]$. Since

$$G = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right],$$

equation (52) gives immediately

$$H = [1 - 2o_A,\ 1 - 2o_B,\ 1 - 2o_C]. \tag{53}$$

Using

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

and

$$\sin B \sin C - \cos A = \cos B \cos C,$$

equation (53) reduces to equation (51). This quick derivation is worth remembering because the Monge point of a simplex will later admit an almost identical Euler-line formula.

A.3 Guy's orthocentric quadrangle

Richard Guy advocated granting the orthocenter the same status as the three vertices and regarding

$$\{A, B, C, H\}$$

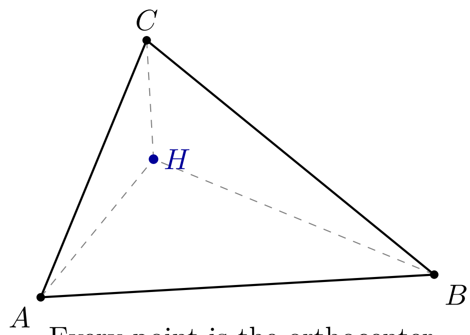
as an *orthocentric quadrangle* [7]. Its defining symmetry is

$$\begin{aligned} H &= \mathrm{Orth}(ABC), \quad A = \mathrm{Orth}(BCH), \\ B &= \mathrm{Orth}(CAH), \quad C = \mathrm{Orth}(ABH). \end{aligned} \tag{54}$$

Thus the designation "center" is relative: every one of the four points is the orthocenter of the triangle formed by the other three.

A second orthocentric quadrangle is fundamental here. If I is the incenter and E_A, E_B, E_C are the excenters of a triangle Δ', then

$$\{I, E_A, E_B, E_C\} \tag{55}$$



Every point is the orthocenter
of the triangle formed by the other three.

Figure 3: The orthocentric-quadrangle viewpoint emphasized by Guy.

is an orthocentric quadrangle. In particular,

$$I = \mathrm{Orth}(E_A E_B E_C). \tag{56}$$

The angle identities

$$\angle E_B E_A E_C = 90^\circ - \frac{A'}{2}, \qquad \angle E_B I E_C = 90^\circ + \frac{A'}{2}$$

make the acute/obtuse dichotomy especially transparent.

B The complete two-dimensional inverse in-excentral problem

B.1 Parents and the EEE/IEE distinction

Definition B.1 (In-ex-parent of a triangle)**.** A triangle Δ' is an *in-ex-parent* of a triangle Δ if the three vertices of Δ are three distinct members of

$$\{I(\Delta'), E_A(\Delta'), E_B(\Delta'), E_C(\Delta')\}.$$

There are only two combinatorial possibilities:

- **EEE**: all three vertices are excenters;
- **IEE**: one vertex is the incenter and the other two are excenters.

Theorem B.2 (Complete planar classification)**.** *Every nonright triangle Δ has a unique in-ex-parent Δ'. More precisely,*

$$\Delta \text{ acute} \iff \Delta \text{ is EEE},$$

and

$$\Delta \text{ obtuse} \iff \Delta \text{ is IEE}.$$

The parent Δ' is the orthic triangle of Δ, with altitude feet taken on extended sidelines when necessary.

Proof. Let H be the orthocenter of $\Delta = ABC$. The four points A, B, C, H form the orthocentric quadrangle equation (54). Its four constituent triangles have the same three altitude feet, so there is a common orthic triangle Δ'. Standard angle chasing shows that the four points A, B, C, H are exactly the incenter and three excenters of Δ'. Hence Δ' is an in-ex-parent of ABC.

If ABC is acute, H is the incenter of Δ' and A, B, C are its three excenters. Equivalently, the excentral-angle formula

$$A = 90^\circ - \frac{A'}{2}, \quad B = 90^\circ - \frac{B'}{2}, \quad C = 90^\circ - \frac{C'}{2}$$

shows that an EEE triangle is necessarily acute and gives the inverse angles

$$A' = 180^\circ - 2A, \quad B' = 180^\circ - 2B, \quad C' = 180^\circ - 2C.$$

If ABC is obtuse, say at A, then A is the point of the orthocentric quadrangle that becomes the incenter of the common orthic triangle; B, C, H are the excenters. Hence ABC is IEE, with the incenter located at its obtuse vertex.

Finally, if ABC were obtained from some other parent, the missing fourth member of its in/excentral quadrangle would have to be H, because the incenter and excenters form an orthocentric quadrangle. The parent is then the common orthic triangle of A, B, C, H, proving uniqueness. In the right-triangle limit, H coincides with the right-angle vertex and the four-point system degenerates. □

Corollary B.3 (The planar Supercenter)**.** *If the Supercenter is defined as the missing in/excentral center of the unique parent, then for every nonright triangle*

$$\boxed{S_2(ABC) = H(ABC).}$$

Thus in two dimensions the new construction produces no new center: it reproduces the orthocenter exactly.

Remark B.4. The planar result is stronger than a mere formula. It says that the inverse in/excentral construction is globally rigid away from the right-triangle boundary. This rigidity is the principal motivation for asking what survives in three dimensions.

C Orthocentric criteria and Euler-line background

A tetrahedron is *orthocentric* when its four altitudes concur. In dimension two this condition is automatic; in dimension three it selects a proper, highly structured family. The following elementary equivalences are useful because each emphasizes a different aspect of that structure.

C.1 Equivalent definitions of an orthocentric tetrahedron

Proposition C.1 (Equivalent characterizations)**.** *Let $T = Q_0Q_1Q_2Q_3$ be a nondegenerate tetrahedron. The following conditions are equivalent.*

(i) *The four altitudes of T are concurrent.*

(ii) *The three pairs of opposite edges are perpendicular.*

(iii) *Any two pairs of opposite edges are perpendicular.*

(iv) *For one—and hence for every—vertex Q_i, the orthogonal projection of Q_i onto the opposite face is the orthocenter of that triangular face.*

(v) *There is a point H and a constant c such that, with $v_i = Q_i - H$,*

$$v_i \cdot v_j = c \qquad (i \neq j). \tag{57}$$

(vi) *There are real numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that*

$$\|Q_i - Q_j\|^2 = \alpha_i + \alpha_j \qquad (i \neq j). \tag{58}$$

When these conditions hold, the point H in (v) *is the common orthocenter.*

Proof. Assume first that the altitudes concur at H. For example, both $Q_0 - H$ and $Q_1 - H$ are perpendicular to the edge Q_2Q_3: the first because Q_2Q_3 lies in the face opposite Q_0, and the second because it lies in the face opposite Q_1. Hence their difference $Q_0 - Q_1$ is perpendicular to $Q_2 - Q_3$. The same argument gives perpendicularity of all three pairs of opposite edges, so (i) implies (ii).

Only two opposite-edge conditions are independent. Indeed, for arbitrary points A, B, C, D,

$$(B - A)\cdot(D - C) - (C - A)\cdot(D - B) + (D - A)\cdot(C - B) = 0. \tag{59}$$

Thus any two vanishing terms force the third to vanish; (ii) and (iii) are equivalent.

Suppose now that the opposite edges are perpendicular and put $Q_0 = 0$, $Q_1 = b$, $Q_2 = c$, $Q_3 = d$. The three perpendicularities give

$$b \cdot c = b \cdot d = c \cdot d =: q.$$

There is a unique vector H satisfying

$$H \cdot b = H \cdot c = H \cdot d = q.$$

These equations say exactly that H lies on the altitude from Q_0 and that $H - Q_i$ is perpendicular to the face opposite Q_i for $i = 1, 2, 3$. Hence (ii) implies (i). They also yield the constant off-diagonal dot-product condition (v). Conversely, subtracting two of the equations in (v) shows immediately that every opposite pair of edges is perpendicular.

For (iv), let P_0 be the orthogonal projection of Q_0 onto the face $Q_1Q_2Q_3$. If T is orthocentric, then the projection property of orthocentric simplices gives that P_0 is the orthocenter of that face [9, 4]. Conversely, if P_0 is the orthocenter of $Q_1Q_2Q_3$, then Q_0P_0 is perpendicular to the face while, for instance, Q_1P_0 is perpendicular to Q_2Q_3. Therefore $Q_0Q_1 = (Q_0 - P_0) + (P_0 - Q_1)$ is perpendicular to Q_2Q_3, and cyclically all opposite pairs are perpendicular.

Finally, from (v),

$$\|Q_i - Q_j\|^2 = \|v_i - v_j\|^2 = (\|v_i\|^2 - c) + (\|v_j\|^2 - c),$$

which is (vi) with $\alpha_i = \|v_i\|^2 - c$. Conversely, the polarization identity

$$2(Q_1 - Q_0) \cdot (Q_3 - Q_2) = \|Q_1 - Q_2\|^2 + \|Q_0 - Q_3\|^2 - \|Q_1 - Q_3\|^2 - \|Q_0 - Q_2\|^2$$

vanishes under equation (58), and the same holds cyclically. Thus (vi) implies (ii). □

Condition (iv) has a useful higher-dimensional form: if F is the facet opposite a vertex V of an orthocentric d-simplex, $d \geq 3$, then F is itself orthocentric and the orthogonal projection of V onto F is the orthocenter of F [9]. This recursive inheritance is one of the structural reasons that orthocentric simplices behave so much like triangles.

Proposition C.2. *For a nonregular orthocentric simplex with interior orthocenter, the incenter lies on its Euler line if and only if its orthocenter barycentric coordinates have at most two distinct values.*

Proof. Let the orthocenter weights be $h_i > 0$, with sum 1. Orthocentric factorization gives $\|g_i\|^2 = \kappa h_i(1-h_i)$ with $\kappa > 0$. Facet hyperareas are proportional to $\|g_i\|$, so incenter weights are proportional to $\sqrt{h_i(1-h_i)}$. The Euler line has barycentrics affine in the h_i, since it passes through H and G. If the incenter is on it, there are constants A, B satisfying $\sqrt{h_i(1-h_i)} = A + Bh_i$ for all i. After squaring,

$$(1+B^2)h_i^2 + (2AB-1)h_i + A^2 = 0,$$

so there are at most two distinct values. Conversely, affine interpolation on two values expresses the incenter weights as an affine function of h_i; normalization places the point on the line through G and H. □

Kites give one possible multiplicity pattern, not the only one. For example, the orthocentric tetrahedron

$$(3,0,5/2), \quad (-3,0,5/2), \quad (0,4,-5/2), \quad (0,-4,-5/2)$$

has $G = 0$, $H = (0,0,7/10)$, $O = -H$, and its incenter is on the z-axis by reflection symmetry. Its facet edge lengths are $(6, \sqrt{50}, \sqrt{50})$ or $(8, \sqrt{50}, \sqrt{50})$, so it has no regular facet and is not a kite.

For comparison with the triangle analogy, the Monge point in dimension d satisfies

$$M_d = \frac{(d+1)G - 2O}{d-1}, \tag{60}$$

so it equals the genuine orthocenter on an orthocentric simplex [2]. Hajja and Martini discuss the broader triangle-like properties of these simplices [9]. One dimension-specific example is the polar-sine angle-sum theorem of Hajja and Hammoudeh [5]: the sum of the appropriately acute/obtuse-adjusted polar-sine angle measures at the four vertices of an orthocentric tetrahedron equals π.

D Vertex purity versus Supercenter sign compatibility

The orthocentric literature suggests a tempting question: does purity of the vertex trihedral angles control the signs in equation (14)? In general it does not.

Proposition D.1 (Logical independence)**.** *For a general tetrahedron, neither of the following conditions implies the other:*

(a) every vertex is pure (strongly acute or strongly obtuse);

(b) $\operatorname{sgn} P_1 = \operatorname{sgn} P_2 = \operatorname{sgn} P_3$.

Proof. For (a) without (b), take

$$Q_0 = (0,0,0), \quad Q_1 = (-2,-2,-1), \quad Q_2 = (-2,-1,2), \quad Q_3 = (-2,1,-1).$$

At the four vertices, the three incident-edge scalar products are respectively

$$(4,3,1), \quad (5,6,3), \quad (5,8,7), \quad (3,5,6),$$

so all four vertices are strongly acute. However

$$(c_{01}, c_{02}, c_{03}, c_{12}, c_{13}, c_{23}) = \left(\frac{1}{36}, \frac{1}{12}, \frac{5}{36}, \frac{7}{108}, \frac{23}{324}, -\frac{1}{108}\right),$$

which gives

$$(P_1, P_2, P_3) = \frac{1}{3888}(-1, 23, 35).$$

The signs are incompatible.

For (b) without (a), take

$$Q_0 = (0,0,0), \quad Q_1 = (-2,-2,-2), \quad Q_2 = (-2,-2,-1), \quad Q_3 = (-2,-1,0).$$

At Q_2 the incident-edge scalar products are $(-1,3,-1)$ and at Q_3 they are $(-1,-1,3)$, so these vertices are mixed and hence not pure. Yet

$$(c_{01}, c_{02}, c_{03}, c_{12}, c_{13}, c_{23}) = \left(\frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, \frac{7}{2}, -\frac{3}{2}, 3\right)$$

and therefore

$$(P_1, P_2, P_3) = \left(\frac{3}{4}, \frac{3}{4}, \frac{7}{4}\right),$$

all positive. □

Orthocentricity is much stronger than either condition separately. By theorem 6.1, an orthocentric tetrahedron satisfies

$$P_1 = P_2 = P_3$$

not merely equality of signs; by [4, Theorem 3.8] its vertex angles satisfy the purity restrictions described earlier.

E Detailed deformation of the opposite quartet

Return to the normal family equations (31) and (34). At $e = f = 0$ the orthocenter barycentrics can be written

$$(w_A, w_B, w_C, w_D) = ((1-\theta)a, (1-\theta)b, (1-\theta)c, \theta), \qquad \theta = \frac{q}{h^2}, \tag{61}$$

where

$$a = \frac{(1-x)r}{2y^2}, \qquad b = \frac{(1+x)r}{2y^2}, \qquad c = \frac{1-x^2}{y^2}. \tag{62}$$

The horizontal perturbation of D changes the barycentrics of its projection on ABC by

$$\delta_A = -\frac{e}{2} - \frac{1-x}{2y}f, \qquad \delta_B = \frac{e}{2} - \frac{1+x}{2y}f, \qquad \delta_C = \frac{f}{y}. \tag{63}$$

Put

$$\rho_A = \frac{\delta_A}{a}, \qquad \rho_B = \frac{\delta_B}{b}, \qquad \rho_C = \frac{\delta_C}{c}, \qquad R = \rho_A + \rho_B + \rho_C,$$

and

$$\mu = \frac{q}{h^2 - q}, \qquad \ell_i = \frac{1}{2}\big[(1-\mu)\rho_i - \mu R\big] \quad (i = A, B, C), \qquad \ell_D = \frac{R}{2}. \tag{64}$$

A direct expansion of the intrinsic branch formula gives

$$\boxed{\sigma_i = w_i(1+\ell_i) + O(\|(e,f)\|^2), \qquad \sum_i w_i \ell_i = 0.} \tag{65}$$

Substituting equation (65) into equation (25) yields all four moving opposite points. If

$$z_s^{(0)} = \sum_i h_{si} w_i, \qquad d_s = \sum_i h_{si} w_i \ell_i,$$

then

$$\boxed{R_s = R_s^{(0)} + \frac{1}{z_s^{(0)}} \left[\sum_i h_{si} w_i \ell_i (Q_i^{(0)} - R_s^{(0)}) + h_{sD} w_D(e,f,0) \right] + O(\|(e,f)\|^2).} \tag{66}$$

Thus the opposite quartet generally moves at first order when orthocentricity is broken.

Nevertheless the reverse operation has *zero defect*. From equation (28),

$$\gamma_s^{(i)} = \frac{h_{si} z_s^{(0)}}{4 w_i} \left(1 + \frac{d_s}{z_s^{(0)}} - \ell_i \right) + O(\|(e,f)\|^2). \tag{67}$$

Substitution of equations (66) and (67) and the identity $\mathsf{H}_4^T \mathsf{H}_4 = 4I$ gives

$$\widehat{A} = A + O(\|(e,f)\|^2), \quad \widehat{B} = B + O(\|(e,f)\|^2), \quad \widehat{C} = C + O(\|(e,f)\|^2), \quad \widehat{D} = D + O(\|(e,f)\|^2).$$

But Corollary 5.2 is stronger: throughout the regular chamber these equalities are exact, not merely first-order consequences.

Corollary E.1 (Stability of the eight-point property)**.** *Let T_0 be a regular orthocentric tetrahedron for which all four real Supercenter branches are finite and its signed parent is nonsingular. Then every sufficiently small deformation T that stays in the same real branch chamber possesses an eight-point configuration*

$$T \cup \mathcal{S}_4(T)$$

with exact bipartite reciprocity

$$\boxed{\mathcal{S}_4(T) = T^*, \qquad \mathcal{S}_4(T^*) = T.}$$

Thus the eight-point property survives to all orders, even though the selected five-point Guy property is destroyed at first order.

Remark E.2 (Singular symmetric cases)**.** The finiteness assumption is essential. For example, for a regular tetrahedron the orthocenter barycentrics are $(1/4, 1/4, 1/4, 1/4)$ and three two-minus signed sums z_s vanish. The corresponding three branches are at infinity, so the finite $4+4$ picture degenerates exactly at that highly symmetric point. A generic nearby tetrahedron moves these branches back to finite positions.

The contrast can now be stated sharply:

$$\boxed{\begin{array}{ll} \text{five-point reciprocity:} & \text{rigid and equivalent to orthocentricity,} \\ \text{eight-point signed reciprocity:} & \text{open-chamber, exact, and involutive.} \end{array}} \tag{68}$$

This is the correct three-dimensional replacement for the single planar in/excentral quadrangle. The four-point object doubles to an eight-point object, but the larger signed structure is substantially more stable than the selected five-point subsystem.

Reproducibility and assistance

The accompanying project contains the editable manuscript, figure sources, and verification scripts. Historical computational records are retained with their provenance and limitations. ChatGPT assisted with mathematical development, proof drafting, code revision, and editing. The author must independently review and take responsibility for the final submitted content.

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