Tetrahedra with Orthogonal and Cevian Euler–Projection Properties

Edge-length classifications with orthocentric, jaw, equifacial, and diametral-mirror components

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Abstract

We classify nondegenerate tetrahedra whose Euler object projects into the Euler object of every face, under either orthogonal projection or central projection from the opposite vertex. The Euler object is the line through the circumcenter and centroid when they differ, and their common point otherwise. The complete orthogonal locus consists of all orthocentric tetrahedra, all equifacial tetrahedra, and jaws having no equilateral face. The central locus contains the same families and additionally the non-equilateral diametral-mirror families, characterized by a circumdiameter and reflection interchanging the remaining two vertices. The distinction comes from face planes containing the circumcenter: their central-projection condition is automatic. Away from this exception, both projection mechanisms give a common six-distance equation involving the opposite altitude foot. The resulting four edge polynomials admit an elementary decomposition. Equilateral faces require a separate incidence argument and force orthocentricity for either all-face property. We give the exact Euclidean realization domain and volume formula for the additional central component, describe intersections and symmetry, and distinguish weak containment from equality of noncollapsed projected lines. All dimensions include scale.

Keywords. Tetrahedron; Euler line; orthogonal projection; cevian projection; orthocentric tetrahedron; diametral mirror.

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1 Introduction

Notation across the series. Throughout the series, $e_{ij} = |V_iV_j|$ denotes an edge length and $d_{ij} = e_{ij}^2$ its square. Dimensions include scale; passing to similarity classes subtracts one. Local six-variable aliases are specified at each metric calculation. In this paper the edge-system variables (x, y, z, u, v, w) mean $(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34})$.

For a triangle, the circumcenter, centroid, and orthocenter lie on the Euler line. A general tetrahedron still has a circumcenter and a centroid, but its four altitudes need not be concurrent. The Monge point supplies a natural replacement for the missing orthocenter: the circumcenter O, centroid G, and Monge point M satisfy

$$M = 2G - O,$$

so that G is the midpoint of OM. The geometry of tetrahedral altitudes and the Monge point has a substantial history; useful modern accounts include Mandan [10], Havlicek–Weiss [9], and Edmonds–Hajja–Martini [3].

Two themes in the literature are especially relevant here. The first is the study of *orthocentric* simplices, in which the altitudes do concur. Orthocentric simplices preserve many of the most attractive features of triangles [7]; their centers and parameterizations are treated systematically in [3, 6]. The second theme is the coincidence of simplex centers. In particular, conditions such as $O = G$, $G = I$, or $O = I$ force significant regularity [2]. For tetrahedra, $O = G$ characterizes the equifacial tetrahedra, also called disphenoids or isosceles tetrahedra; see Hajja–Walker [8].

Projection provides a different bridge between planar and spatial centers. Orthogonal projection sends the tetrahedral circumcenter to the circumcenter of a face. Cevian projection from the opposite vertex sends the tetrahedral centroid to the face centroid. These two facts suggest parallel questions: when does either projection send the entire tetrahedral Euler object into the face Euler line, simultaneously for all four faces? These four-face conditions do not appear to have been isolated as named classes in the cited literature.

The terminology *jaw* was introduced by Hajja, Hammoudeh, Hayajneh, and Martini in their study of concurrence of tetrahedral cevians associated with triangle centers [5]. That paper also gave the four-equal-edge description and a lifted-rhombus construction. The present problem is different: it projects one spatial Euler object onto each face rather than asking four cevians to prescribed facial centers to concur. The reappearance of jaws here is therefore a classification result, not a new definition of the family.

The present article is the projection-theoretic $k = 2$ member of a four-paper program. A companion article introduces the four-dimensional power–k families

$$\mathcal{P}_k : \qquad e_{ij}^k = p_i + p_j$$

and their Cevian–pedal theory (companion P); its member $\mathcal{P}_2$ is the full orthocentric family. A second companion article studies lower-dimensional face constraints and analytic Z-cevian dimension obstructions (companion D). The skeletal-centroid projection problem, whose maximal cevian component is $\mathcal{P}_1$, is treated separately in (companion S). These papers concern different spatial compatibility mechanisms—concurrence, face constraints, and projection— but share the same six-edge parameter space.

The orthogonal classification is the union of:

(i) the four-dimensional orthocentric family;

(ii) the open parts, having no equilateral face, of three labeled, mutually equivalent, three-dimensional jaw components, each characterized by a cycle of four equal edges;

(iii) the three-dimensional equifacial component, on which the tetrahedral Euler line degenerates to a point.

The cevian classification contains these components and, in addition, the non-equilateral parts of six labeled diametral-mirror components, one for each choice of a distinguished edge. Their generic member has a single reflection plane. The additional family arises precisely from the exceptional case in which the circumcenter belongs to two face planes. The boundary omitted by the phrase "non-equilateral" is not a new component: we prove that either projection property, in the presence of an equilateral face, forces orthocentricity.

Thus the largest component is four-dimensional, not five-dimensional, for either notion of projection. All dimensions in this paper include scale; one dimension is subtracted when tetrahedra are considered up to similarity.

Companion manuscripts. The unpublished companion papers by the present author are designated (P) *Power-k tetrahedra: a unified Cevian–pedal theory of four-parameter tetrahedral families*; (D) *Dimensions of tetrahedral Z-cevian families: face constraints, jaws, tripods, and odd-dimensional obstructions*; (S) *Skeletal-centroid projection properties of tetrahedra: circumscriptible families, jaws, and a symmetry principle.* All are version 2, dated 6 September 2026; these are local version identifiers, not public repository records. Results needed for the present proofs are stated and proved here.

2 Euler lines and their projections

Let $T = ABCD$ be a non-coplanar tetrahedron. For a face F, write O_F, G_F, H_F for its circumcenter, centroid, and orthocenter, and write π_F for orthogonal projection onto $\operatorname{aff}(F)$. If X_F is the vertex opposite F, central projection of the tetrahedral Euler object will be defined directly as a line-object, rather than point by point.

Definition 2.1 (Euler objects and their projections). (a) The *Euler object* of T is

$$\mathcal{E}_T = \begin{cases} \operatorname{aff}\{O, G\}, & O \neq G, \\ \{O\}, & O = G. \end{cases}$$

The Euler object of a triangular face is

$$\mathcal{E}_F = \begin{cases} \operatorname{aff}\{O_F, G_F\}, & O_F \neq G_F, \\ \{O_F\}, & O_F = G_F. \end{cases}$$

Thus $\mathcal{E}_F$ is a line for a non-equilateral face and a point for an equilateral face.

(b) The *cevian image* of $\mathcal{E}_T$ from the opposite vertex is

$$\rho_F(\mathcal{E}_T) := \operatorname{aff}(X_F, \mathcal{E}_T) \cap \operatorname{aff}(F). \tag{1}$$

Here $\operatorname{aff}(X_F, \mathcal{E}_T)$ is a plane when $\mathcal{E}_T$ is a line not containing X_F, and otherwise is a line. The intersection in (1) is always nonempty: $G \in \mathcal{E}_T$, and the median $X_F G$ meets F at G_F. Consequently the cevian image is either a line or the single point G_F; no affine "point at infinity" convention is required.

(c) The tetrahedron has the *orthogonal Euler–projection property* (OEPP) if, for every face F,

$$\pi_F(\mathcal{E}_T) \subseteq \mathcal{E}_F.$$

It has the *cevian Euler–projection property* (CEPP) if

$$\rho_F(\mathcal{E}_T) \subseteq \mathcal{E}_F \qquad \text{for every face } F.$$

(d) The corresponding property is called *strong* when $\mathcal{E}_T$ and all four $\mathcal{E}_F$ are lines and every displayed containment is equality. Thus a strong property excludes both a degenerate Euler object and a collapsed projected line.

The line-object formulation is important in two different degeneracies. A non-coplanar tetrahedron can have $O = G$, and a perfectly nondegenerate face can be equilateral. In either event the relevant Euler object is a point rather than a line.

Lemma 2.2 (Projection identities). *Let $F = ABC$, and let*

$$P = \pi_F(D)$$

be the foot of the altitude from D to ABC. Then

$$\pi_F(O) = O_F, \qquad \pi_F(G) = \frac{3G_F + P}{4}, \qquad \pi_F(M) = \frac{H_F + P}{2}. \tag{2}$$

If F is non-equilateral, the orthogonal one-face condition is

$$\pi_F(\mathcal{E}_T) \subseteq \mathcal{E}_F \quad \Longleftrightarrow \quad P \in \mathcal{E}_F. \tag{3}$$

If $O \neq G$, the containment in (3) is an equality unless OG is perpendicular to F. If F is equilateral, then

$$\pi_F(\mathcal{E}_T) \subseteq \mathcal{E}_F \quad \Longleftrightarrow \quad P = G_F. \tag{4}$$

Proof. Every vertex of F has the same distance from O. Orthogonal projection onto F removes the same normal component from the three vectors joining O to the vertices of F; hence $\pi_F(O)$ is equidistant from A, B, C, and therefore equals O_F.

Since $G = (A + B + C + D)/4$ and orthogonal projection is affine,

$$\pi_F(G) = \frac{A + B + C + P}{4} = \frac{3G_F + P}{4}.$$

Finally, $M = 2G - O$, while the triangular Euler relation is $H_F = 3G_F - 2O_F$. Therefore

$$\pi_F(M) = 2\pi_F(G) - \pi_F(O) = \frac{3G_F + P}{2} - O_F = \frac{H_F + P}{2}.$$

If F is non-equilateral, then $\mathcal{E}_F = \operatorname{aff}\{O_F, G_F, H_F\}$, and equations (2) show that all projected tetrahedral Euler centers lie on $\mathcal{E}_F$ exactly when P does. If F is equilateral, then $O_F = G_F = H_F$ and $\pi_F(O) = O_F$; hence the image is contained in the single point $\mathcal{E}_F$ exactly when $\pi_F(G) = (3G_F + P)/4 = G_F$, equivalently $P = G_F$. □

The cevian analogue is almost the same, but it has one geometrically important exception.

Lemma 2.3 (Cevian one-face dichotomy)**.** *Let $F = ABC$ be non-equilateral, let D be the opposite vertex, and let $P = \pi_F(D)$. Then*

$$\rho_F(\mathcal{E}_T) \subseteq \mathcal{E}_F \iff (P \in \mathcal{E}_F) \text{ or } (O \in \operatorname{aff}(F)). \tag{5}$$

Moreover,

$$G_F \in \rho_F(\mathcal{E}_T).$$

If $O \notin \operatorname{aff}(F)$, the cevian and orthogonal one-face conditions are therefore equivalent. If $O \in \operatorname{aff}(F)$, then $O = O_F$ and the cevian condition is automatic.

Proof. The identity

$$G = \frac{3G_F + D}{4}$$

puts D, G, G_F on one line; since $G \in \mathcal{E}_T$, the cevian image contains G_F. Choose coordinates in which $\operatorname{aff}(F)$ is $z = 0$, and write

$$D = (P, k), \qquad O = (O_F, h).$$

The cevian image of $\mathcal{E}_T$ is the intersection with F of the affine hull generated by D, O, G_F. It already contains G_F. If the image is a line, it is contained in $\mathcal{E}_F = \operatorname{aff}\{G_F, O_F\}$ exactly when O_F belongs to that affine hull. If the image collapses to the point G_F, the containment is automatic; in this case D, O, G_F are collinear and the determinant below vanishes automatically. In both cases the governing scalar triple product is therefore

$$\det(D - G_F, O - G_F, O_F - G_F) = -h \det\nolimits_2(P - G_F, O_F - G_F),$$

which vanishes precisely when $h = 0$, meaning $O \in \operatorname{aff}(F)$, or when P, G_F, O_F are collinear, meaning $P \in \mathcal{E}_F$. If $h = 0$, the point O is equidistant from A, B, C and lies in their plane, so it equals O_F. □

Lemma 2.4 (Equilateral one-face condition)**.** *Let F be equilateral and let X_F be its opposite vertex. Then*

$$\rho_F(\mathcal{E}_T) \subseteq \mathcal{E}_F$$

holds if and only if either $O = G$, or $X_F \in \mathcal{E}_T$.

Proof. Now $\mathcal{E}_F = \{G_F\}$, and the median identity shows that X_F, G, G_F are collinear. If $O = G$, the cevian image of the single point O is therefore G_F. If $O \neq G$, central projection maps the line $\mathcal{E}_T$ to the single point G_F exactly when its center X_F lies on $\mathcal{E}_T$. Otherwise $\operatorname{aff}(X_F, \mathcal{E}_T)$ is a plane and its intersection with F is a line, which cannot be contained in $\{G_F\}$. □

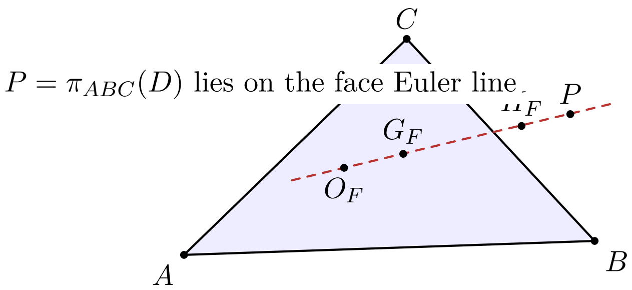

Figure 1: The ordinary one-face condition. Orthogonal projection, and cevian projection when $O \notin \mathrm{aff}(ABC)$, are compatible with the face Euler line precisely when the altitude foot P lies on that line.

3 A metric criterion for the ordinary branch

The projection condition can be expressed without coordinates of points or centers.

Lemma 3.1 (Six-distance criterion)**.** *Let UVW be a non-equilateral triangular face and X the opposite vertex. Put*

$$UV^2 = a, \quad UW^2 = b, \quad VW^2 = c,$$

and

$$XU^2 = d_0, \quad XV^2 = d_1, \quad XW^2 = d_2.$$

The projection of X onto UVW lies on the Euler line of UVW if and only if

$$\boxed{(b-a)d_0 + (a-c)d_1 + (c-b)d_2 = 0.} \tag{6}$$

Proof. Take U as origin and use the ordered basis $\mathbf{v} = V - U, \mathbf{w} = W - U$ in the face plane. Its Gram matrix is

$$K = \begin{pmatrix} a & (a+b-c)/2 \\ (a+b-c)/2 & b \end{pmatrix}.$$

If p, o, h are the coordinate columns of the altitude foot P, the face circumcenter O_F, and the face orthocenter H_F, then the polarization identity gives

$$p = K^{-1}\frac{1}{2}\begin{pmatrix} d_0 + a - d_1 \\ d_0 + b - d_2 \end{pmatrix}, \qquad o = K^{-1}\frac{1}{2}\begin{pmatrix} a \\ b \end{pmatrix}.$$

The vector identity $H_F = U + V + W - 2O_F$ gives

$$h = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2o.$$

Now P, O_F, H_F are collinear precisely when $\det(p - o, h - o) = 0$. Substitution of the displayed columns, followed by multiplication by the nonzero factor $\det K$, reduces the determinant to (6). □

Remark 3.2. For an equilateral face one has $a = b = c$, so the left-hand side of (6) vanishes identically for every opposite vertex. It must not then be mistaken for the geometric condition: the facial Euler object has collapsed to one point, and the correct requirements are (4) and Lemma 2.4. This is why the equilateral boundary is treated separately below.

4 The edge equations

Use the squared-edge notation

$$\begin{aligned} x &= AB^2, \quad y = AC^2, \quad z = AD^2, \\ u &= BC^2, \quad v = BD^2, \quad w = CD^2. \end{aligned} \tag{7}$$

Applying Lemma 3.1 to the four faces yields

$$E_1 = (y-x)z + (x-u)v + (u-y)w = 0, \tag{8a}$$
$$E_2 = (z-x)y + (x-v)u + (v-z)w = 0, \tag{8b}$$
$$E_3 = (z-y)x + (y-w)u + (w-z)v = 0, \tag{8c}$$
$$E_4 = (v-u)x + (u-w)y + (w-v)z = 0. \tag{8d}$$

Although there are four equations in six variables, they are not independent. In fact,

$$E_1 - E_2 + E_3 - E_4 = 0, \tag{9}$$

so any three imply the fourth. Their common zero set has several components.

Definition 4.1 (Jaw)**.** Following Hajja, Hammoudeh, Hayajneh, and Martini, who introduced the term and the underlying family in [5], a tetrahedron is called a *jaw* if its vertices can be cyclically labeled so that

$$AB = BC = CD = DA.$$

Thus a jaw has a cycle of four equal edges; the two remaining, opposite edges AC and BD will be called its diagonals.

Remark 4.2 (Power–2 interpretation). The principal component $\mathcal{O}$ is exactly the power–2 family $\mathcal{P}_2$ of (companion P). Indeed, the equal-opposite-sum relations

$$z + u = y + v = x + w$$

are equivalent to the existence of signed numbers p_A, p_B, p_C, p_D such that

$$AB^2 = p_A + p_B, \quad AC^2 = p_A + p_C, \quad \ldots, \quad CD^2 = p_C + p_D.$$

For example

$$p_A = \frac{x+y-u}{2}, \qquad p_B = \frac{x+u-y}{2}, \qquad p_C = \frac{y+u-x}{2}, \qquad p_D = \frac{z+v-x}{2},$$

where the equal-opposite-sum relations make the analogous formulas for p_D agree. Allowing signs is essential: it is what makes $\mathcal{P}_2$ the full orthocentric family rather than only an acute-type positive-parameter subfamily.

Theorem 4.3 (Orthogonal classification)**.** *Let $T = ABCD$ be a non-coplanar tetrahedron with no equilateral face. Then T has* OEPP *if and only if at least one of the following holds:*

(i) *T is orthocentric;*

(ii) *T is a jaw;*

(iii) *T is equifacial.*

More explicitly, in the notation (7)*, the solution set of* (8) *is the union of*

$$
\begin{aligned}
\mathcal{O}: &\quad z+u=y+v=x+w, &(10)\\
\mathcal{J}_1: &\quad x=u=w=z, &(11)\\
\mathcal{J}_2: &\quad y=u=v=z, &(12)\\
\mathcal{J}_3: &\quad x=y=v=w, &(13)\\
\mathcal{D}: &\quad x=w, \quad y=v, \quad z=u. &(14)
\end{aligned}
$$

Here $\mathcal{O}$ is the orthocentric component, the three $\mathcal{J}_i$ are the three labeled jaw components, and $\mathcal{D}$ is the equifacial or disphenoid component.

On $\mathcal{D}$, $O = G = M$, so the tetrahedral Euler object is a point. The strong orthogonal property holds exactly for the orthocentric and jaw members satisfying

$$O \neq G \qquad \textit{and} \qquad \mathcal{E}_T \not\subset \operatorname{aff}(F) \quad \textit{for every face } F.$$

The first condition excludes a collapsed spatial Euler object; the second excludes a collapsed orthogonal image.

Proof. The four projection conditions are exactly (8). Introduce the two orthocentric defects

$$\alpha = z+u-y-v, \qquad \beta = z+u-x-w. \tag{15}$$

They have an immediate geometric interpretation:

$$2\,\overrightarrow{AB}\cdot\overrightarrow{CD} = \alpha, \qquad 2\,\overrightarrow{AC}\cdot\overrightarrow{BD} = \beta,$$

up to reversing the orientation of an opposite edge, and the third opposite-edge scalar product is represented by $\beta - \alpha$. Consequently,

$$\alpha = \beta = 0$$

is equivalent to all three pairs of opposite edges being perpendicular, which is the standard metric characterization of an orthocentric tetrahedron [1, 3].

Solve (15) for

$$v = z+u-y-\alpha, \qquad w = z+u-x-\beta.$$

After this substitution, the four equations become

$$
\begin{aligned}
E_1 &= \alpha(u-x)+\beta(y-u),\\
E_2 &= \alpha(\beta+x-z)+\beta(y-u),\\
E_3 &= \alpha(\beta+x-u)+\beta(y-z),\\
E_4 &= \alpha(z-x)+\beta(y-z).
\end{aligned}
$$

We now make an elementary case distinction.

Case 1: $\alpha = \beta = 0$. This is precisely the orthocentric component $\mathcal{O}$.

Case 2: $\alpha = 0$, $\beta \neq 0$. The equations $E_1 = E_3 = 0$ give

$$y = u = z.$$

The definition of α then gives $v = z$. Hence

$$y = u = v = z,$$

which is the jaw component $\mathcal{J}_2$.

Case 3: $\beta = 0$, $\alpha \neq 0$. Similarly, $E_1 = E_2 = 0$ give

$$x = u = z,$$

and the definition of β gives $w = x$. Thus

$$x = u = w = z,$$

which is $\mathcal{J}_1$.

Case 4: $\alpha\beta \neq 0$. Two useful differences are

$$E_2 - E_1 = -\alpha(w - x), \qquad E_1 - E_4 = (\alpha - \beta)(u - z).$$

Therefore $w = x$, and either $\alpha = \beta$ or $u = z$.

If $\alpha = \beta$, then $E_1 = 0$ yields $y = x$; the definitions of α, β then give $v = x$. Hence

$$x = y = v = w,$$

the third jaw component $\mathcal{J}_3$.

It remains to take $u = z$ with $\alpha \neq \beta$. Since $w = x$,

$$\beta = 2(u - x).$$

Now $E_1 = 0$ simplifies to

$$(u - x)(y - v) = 0.$$

Because $\beta \neq 0$, one has $u \neq x$, and therefore $y = v$. Together with $w = x$ and $u = z$, this is

$$x = w, \qquad y = v, \qquad z = u,$$

the disphenoid component $\mathcal{D}$.

This proves that there are no further algebraic components. Conversely, direct substitution shows that each of (10)–(14) satisfies all four equations.

Finally, opposite-edge equality in (14) makes all four faces congruent. Conversely, equifaciality forces the three pairs of opposite edges to be equal. It is classical that a tetrahedron satisfies $O = G$ exactly in this case [8, 2]. Thus $\mathcal{D}$ is precisely the degenerate-Euler-line component. When $O \neq G$, Lemma 2.2 turns the collinearity conditions into equality of lines unless $\mathcal{E}_T$ is perpendicular to the face, in which case its orthogonal image collapses. This proves the strong statement as well. □

5 The cevian classification

Lemma 2.3 shows why the two projection notions are close but not identical. A face produces the same equation $E_i = 0$ unless it contains the tetrahedral circumcenter. At most two face planes can contain O: three face planes meet at a vertex, and a circumcenter cannot be a vertex of a non-coplanar tetrahedron.

Definition 5.1 (Diametral-mirror tetrahedron)**.** A tetrahedron is *diametral-mirror* with distinguished edge AB if

$$AC = AD, \qquad BC = BD, \qquad AB^2 = AC^2 + BC^2. \tag{16}$$

The six possible choices of distinguished edge give six labeled components.

Theorem 5.2 (Cevian classification)**.** *Let $T = ABCD$ be a non-coplanar tetrahedron with no equilateral face. Then T has* CEPP *if and only if it is one of the following:*

(i) *orthocentric;*

(ii) *a jaw;*

(iii) *equifacial;*

(iv) *diametral-mirror, for one of its six edges.*

For the distinguished edge AB, the additional component is

$$\mathcal{M}_{AB}: \qquad y = z, \qquad u = v, \qquad x = y + u. \tag{17}$$

The strong cevian property holds precisely for members of these families with $O \neq G$ and with no vertex lying on $\mathcal{E}_T$.

Proof. If O lies in no face plane, all four conditions are the ordinary equations (8), so Theorem 4.3 applies. If O lies in exactly one face plane, the other three ordinary equations hold; the syzygy (9) supplies the fourth, and the same theorem again applies.

Suppose now that O lies in two face planes. Relabel so these are ABC and ABD. Their intersection is the line AB, and a point on this line equidistant from A and B is their midpoint. Thus O is the midpoint of AB, and AB is a diameter of the circumsphere. Thales' theorem in the two face planes gives

$$x = y + u = z + v, \tag{18}$$

or $u = x - y$ and $v = x - z$. The cevian conditions for ABC and ABD are automatic. Substitution into the remaining two ordinary equations gives

$$E_3 = (z - y)(z + y - w), \tag{19}$$
$$E_4 = (y - z)(2x - y - z - w). \tag{20}$$

If $y = z$, then (18) gives $u = v$, and we obtain $\mathcal{M}_{AB}$. If $y \neq z$, the two other factors give $w = y + z = x$, followed by $u = z$ and $v = y$. This is the equifacial component $x = w, y = v, z = u$, already present in the list. There can be no third face plane through O, so the classification is complete.

Conversely, the old three families satisfy all four ordinary conditions. A member of $\mathcal{M}_{AB}$ has O in the two face planes through AB, and direct substitution shows that the two remaining equations vanish. Lemma 2.3 therefore proves CEPP. Finally, a cevian image of a nondegenerate line collapses exactly when its center of projection, the opposite vertex, lies on that line. This gives the strong statement. □

6 Completion at equilateral faces

The preceding classifications assumed that every facial Euler object was a line. The missing boundary is rigid enough to admit a complete treatment.

Theorem 6.1 (Equilateral-face rigidity)**.** *Let T be a non-coplanar tetrahedron having at least one equilateral face. Then the following are equivalent:*

(i) *T has* OEPP*;*

(ii) *T has* CEPP*;*

(iii) *T is orthocentric.*

Proof. Let the equilateral face be $F = ABC$, put its common center $Q = O_F = G_F = H_F$ at the origin, and take $\text{aff}(F)$ as the horizontal plane. Write

$$D = (p, h), \qquad h \neq 0,$$

where $p \in \text{aff}(F)$, and let R be the circumradius of ABC. Since $\pi_F(O) = Q$, while the centroid is the average of the four vertices, there is a real number t such that

$$O = (0, t), \qquad G = \frac{1}{4}(p, h), \qquad t = \frac{\|p\|^2 + h^2 - R^2}{2h}. \tag{21}$$

Suppose first that T has OEPP. Because $\mathcal{E}_F = \{Q\}$, equation (4) gives $p = 0$. Thus D lies on the normal through the center of the equilateral face and

$$DA = DB = DC.$$

For example, AB is perpendicular both to the face median CQ and to the normal DQ, hence $AB \perp CD$. The two other pairs of opposite edges are treated cyclically. Therefore T is orthocentric.

Suppose next that T has CEPP. If $O = G$, comparison of the horizontal components in (21) gives $p = 0$, and the preceding argument applies. Let $O \neq G$. Lemma 2.4 says that $D \in \mathcal{E}_T$. But D, G, Q are collinear, whereas O lies on the normal through Q. Hence either $p = 0$, or $O = Q$. Again the first alternative is orthocentric.

Assume for a contradiction that $p \neq 0$ and $O = Q$. All four vertices then lie on the sphere of radius R centered at Q. None of the three lateral faces is equilateral. Indeed, after division of the vertex vectors by R, the equilateral base satisfies $a + b + c = 0$ and $a \cdot b = b \cdot c = c \cdot a = -1/2$. If, for example, ABD were equilateral, then also $a \cdot d = b \cdot d = -1/2$, and

$$\|a + b + d\|^2 = 0.$$

Thus $d = -(a + b) = c$, contradicting non-coplanarity.

Moreover, Q belongs to no lateral face plane: such a plane meets the base plane in one side of ABC, and the center Q lies on no side. Lemma 2.3 therefore supplies the ordinary edge equation for each lateral face. The base equation vanishes because its three squared side lengths are equal. Hence all four equations (8) hold. The algebraic decomposition proved in Theorem 4.3 uses no non-equilateral hypothesis, so the squared edge lengths belong to one of $\mathcal{O}, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{D}$.

Put $x = y = u = s$ for the equilateral base. On $\mathcal{O}$ one gets $z = v = w$, which forces $p = 0$. The component $\mathcal{D}$ makes all six edges equal and again forces $p = 0$. Finally, $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ respectively make ACD, ABD, BCD equilateral, contrary to the preceding paragraph. Thus the alternative $p \neq 0$ is impossible, proving that either projection property implies orthocentricity.

Conversely, in an orthocentric tetrahedron the altitude from each vertex meets the opposite face at its orthocenter. This proves the ordinary one-face condition on every non-equilateral face. On an equilateral face the altitude foot is its common center, so the orthogonal image is the facial Euler point. The corresponding vertex, the tetrahedral circumcenter, and the centroid lie on the same normal axis, so Lemma 2.4 gives the cevian condition as well. Hence an orthocentric tetrahedron with an equilateral face has both properties. □

Corollary 6.2 (Complete classifications)**.** *For arbitrary non-coplanar tetrahedra, with no restriction on the faces, the* OEPP *locus is the union of*

(i) *all orthocentric tetrahedra;*

(ii) *all equifacial tetrahedra;*

(iii) *jaws having no equilateral face.*

The CEPP *locus is the union of the same three sets and the diametral-mirror tetrahedra having no equilateral face.*

Proof. Apply Theorems 4.3 and 5.2 when every face is non-equilateral, and Theorem 6.1 otherwise. □

Corollary 6.3. *Neither projection-property locus has a five-dimensional component. Including scale, the maximum dimension is four. The cevian locus adds six labeled three-dimensional components to the orthogonal locus.*

Proof. The orthocentric equations in (10) impose two independent conditions on the six edge lengths. Each jaw component, the disphenoid component, and each component (16) impose three. The open realizability inequalities do not alter these local dimensions. □

Remark 6.4. The absence of a five-dimensional component here follows directly from the explicit projection classification. This is logically distinct from the nonzero-weight analytic no-5 obstruction for proper Z-cevian families proved in (companion D); the two results concern different compatibility mechanisms.

7 Diametral mirrors and mirrored wings

The additional cevian component belongs to a familiar general symmetry class, but the circumdiameter constraint selects a sharper subfamily.

Definition 7.1 (Mirrored wing)**.** A tetrahedron has *mirrored wings* about the edge AB if

$$AC = AD, \qquad BC = BD. \tag{22}$$

Equivalently, it has a reflection symmetry fixing A, B and interchanging C, D.

Indeed, the two equalities say that both A and B lie in the perpendicular-bisector plane of CD. That plane contains AB, and reflection in it interchanges C, D. Conversely, such a reflection immediately gives the two equal pairs. Thus the phrase "wings" refers to the congruent triangles ABC and ABD, hinged along the fixed edge AB.

Proposition 7.2 (Characterization of the diametral-mirror family)**.** *For a non-coplanar tetrahedron $ABCD$, the following are equivalent.*

(a) *It is diametral-mirror with distinguished edge AB.*

(b) *It has mirrored wings about AB, and AB is a diameter of its circumsphere.*

(c) *Reflection in a plane through AB exchanges C, D, and*

$$\angle ACB = \angle ADB = 90^\circ.$$

(d) *After a rigid motion it has the normal form*

$$\begin{aligned} A &= (-a, 0, 0), \quad B = (a, 0, 0), \\ C &= (p, q, 0), \qquad D = (p, q\cos\theta, q\sin\theta), \end{aligned} \tag{23}$$

where $a > 0$, $p^2 + q^2 = a^2$, $q > 0$, and $\sin\theta \neq 0$.

Its circumcenter is the midpoint of AB, and the family is three-dimensional when scale is included.

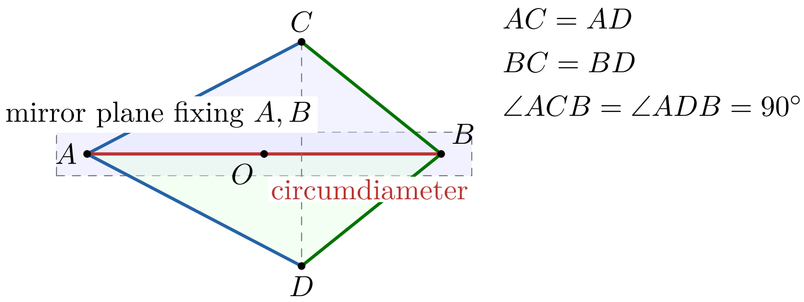

Figure 2: Schematic projection of a diametral-mirror tetrahedron, a right-angled member of the mirrored-wing family. The shaded reflection plane fixes the red circumdiameter AB and exchanges the two wings.

Proof. Condition (a) contains (22); hence there is a reflection fixing AB and swapping C, D. The last equality in (16), together with the cosine rule, says that ABC is right at C. Its circumcenter is therefore the midpoint of AB. The same reflection sends this statement to the face ABD, proving (b) and (c). Conversely, either (b) or (c) gives Thales' equality $AB^2 = AC^2 + BC^2$ and the two mirrored-wing equalities.

For the normal form, place the midpoint of AB at the origin and put AB on the first coordinate axis. The equal-pair conditions force C, D to have the same first coordinate p and the same distance q from that axis. Rotate about AB to place C in the xy-plane. The diameter condition says that C, D lie on the sphere of radius a, which is exactly $p^2 + q^2 = a^2$. This proves (d), and the reverse implication is immediate. The free parameters may be taken as a, p, θ, proving the dimension statement. □

Remark 7.3 (Literature and terminology). Wirth and Dreiding give the edge criterion $AC = AD, BC = BD$ for a mirror reflection interchanging C, D, as part of their classification of tetrahedral symmetries by edge and angle data [11]. Thus the general mirrored-wing family is present in the literature under the broader heading of mirror-symmetric tetrahedra. We have not located a peer-reviewed treatment that isolates the additional right-angle condition $AB^2 = AC^2 + BC^2$ as a separately named class. The term *diametral mirror* is used here for precisely this codimension-one subfamily of mirrored wings.

8 The equifacial component and symmetry

The three-dimensional equifacial component is symmetric in a stronger way than a generic jaw. A convenient coordinate model is

$$\begin{aligned} A &= (p, q, r), & B &= (p, -q, -r), \\ C &= (-p, q, -r), & D &= (-p, -q, r), \end{aligned} \tag{24}$$

with $pqr \neq 0$. Opposite edges have equal lengths, and the coordinate axes are three mutually perpendicular half-turn symmetry axes. Their intersection is

$$O = G = M.$$

For an equifacial tetrahedron this point is also the incenter; see [8].

Remark 8.1 (Axis symmetry versus central symmetry). A generic jaw has one twofold rotation axis, whereas a generic equifacial tetrahedron has three mutually perpendicular twofold rotation axes. The latter axes meet at the common center $O = G = M$, but this point is *not* a center of inversion symmetry of the tetrahedron. In fact, no non-coplanar tetrahedron can be centrally symmetric: four vertices paired as $\{\pm p, \pm q\}$ would lie in the plane spanned by p, q. Thus "central" should refer here to the coincidence of centers, not to point-reflection symmetry.

For completeness, the degeneracy criterion has a short proof. Put the centroid at the origin and denote the vertex vectors by $a_1, \ldots, a_4$, so $\sum a_i = 0$. The sum of squared lengths of the three edges incident with a_i is

$$\sum_{j \neq i} \|a_i - a_j\|^2 = 4\|a_i\|^2 + \sum_{j=1}^{4} \|a_j\|^2.$$

Hence G is the circumcenter precisely when these four incident edge-square sums are equal. In the notation (7), solving

$$\begin{aligned} x + y + z &= x + u + v \\ &= y + u + w \\ &= z + v + w \end{aligned}$$

gives

$$x = w, \qquad y = v, \qquad z = u.$$

This is exactly the equifacial condition.

9 Examples

Example 9.1 (Orthocentric but not a jaw)**.** Let

$$A = (0, 0, 0), \quad B = (2, 0, 0), \quad C = (0, 3, 0), \quad D = (0, 0, 4).$$

The squared edge lengths are

$$(x, y, z, u, v, w) = (4, 9, 16, 13, 20, 25),$$

and

$$x + w = y + v = z + u = 29.$$

Thus the tetrahedron is orthocentric. It has no cycle of four equal edges and is not equifacial.

Example 9.2 (A non-orthocentric jaw)**.** Let

$$A = (-1, 0, 0), \quad B = (0, 2, 1), \quad C = (1, 0, 0), \quad D = (0, -2, 1).$$

Then

$$AB = BC = CD = DA = \sqrt{6}, \qquad AC = 2, \qquad BD = 4.$$

It is therefore a jaw. It is not orthocentric, since

$$\overrightarrow{AB} \cdot \overrightarrow{CD} = (1, 2, 1) \cdot (-1, -2, 1) = -4 \neq 0.$$

Its centroid and circumcenter are

$$G = (0, 0, \tfrac{1}{2}), \qquad O = (0, 0, 2),$$

so its Euler line is nondegenerate and coincides with the vertical half-turn axis. This gives an exact counterexample to the assertion that all-face Euler projection forces orthocentricity.

Example 9.3 (A diametral mirror that is neither orthocentric nor a jaw)**.** Let

$$A = (-1, 0, 0), \quad B = (1, 0, 0), \quad C = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0), \quad D = (\tfrac{1}{2}, 0, \tfrac{\sqrt{3}}{2}).$$

Its squared edge lengths are

$$(x, y, z, u, v, w) = (4, 3, 3, 1, 1, \tfrac{3}{2}).$$

Thus

$$AC = AD = \sqrt{3}, \qquad BC = BD = 1, \qquad AB^2 = AC^2 + BC^2 = 4.$$

The origin is the circumcenter and the midpoint of the distinguished edge AB; reflection in the plane $y = z$ interchanges C, D. The tetrahedron is not orthocentric, because

$$AB^2 + CD^2 = \tfrac{11}{2} \neq 4 = AC^2 + BD^2 = AD^2 + BC^2.$$

It has no four-cycle of equal edges and is not equifacial. Hence it has CEPP but not OEPP, proving that the two classifications are genuinely different.

Example 9.4 (Equifacial with degenerate Euler line)**.** Take

$$\begin{aligned} A &= (1, 2, 3), & B &= (1, -2, -3), \\ C &= (-1, 2, -3), & D &= (-1, -2, 3). \end{aligned}$$

The squared edge lengths are

$$(x, y, z, u, v, w) = (52, 40, 20, 20, 40, 52).$$

Thus

$$x = w, \qquad y = v, \qquad z = u,$$

so the tetrahedron is equifacial. Its volume is nonzero, but

$$O = G = M = (0, 0, 0).$$

Under orthogonal projection the Euler object goes to the face circumcenter; under cevian projection it goes to the face centroid. Both points lie on the face Euler line.

10 Dimensions, intersections, and symmetry strata

A labeled tetrahedron up to Euclidean congruence is locally described by its six edge lengths, subject to the open face and Cayley–Menger inequalities. Therefore linear equalities among squared edge lengths give the dimensions shown in Table 1.

The three jaw equations $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are different in a labeled edge space, but vertex relabeling permutes them. They therefore represent one geometric family.

The components meet along more symmetric subfamilies. In $\mathcal{J}_1$, for example,

$$x = u = w = z = s,$$

while y and v are the squared diagonal lengths. This jaw is equifacial exactly when

$$y = v.$$

Geometrically, its two diagonals then have equal length; algebraically, it lies in $\mathcal{J}_1 \cap \mathcal{D}$. At this intersection the Euler line degenerates. The regular tetrahedron lies in all three orthogonal families, but not in a diametral-mirror component: none of its faces is right. Within the nondegenerate Euclidean edge cone, a diametral-mirror component meets the jaw family but meets neither the orthocentric nor the equifacial family. This distinction is essential: algebraic intersections on the degenerate boundary are not intersections of tetrahedral families.

Table 1: Components of the two Euler–projection loci. Dimensions include scale. A checkmark means that the generic member belongs to the indicated locus.

Component	Generic metric condition	OEPP	CEPP	Dimension	Generic symmetry
All tetrahedra	six realizable edge lengths	–	–	6	trivial
Orthocentric	$z+u=y+v=x+w$	✓	✓	4	generally trivial
Jaw	one four-cycle of equal edges; no equilateral face	✓	✓	3	one half-turn axis; two mirror planes
Equifacial	$x=w,\ y=v,\ z=u$	✓	✓	3	three half-turn axes
Diametral mirror about AB	$y=z,\ u=v,\ x=y+u$; no equilateral face	–	✓	3	one mirror plane

Proposition 10.1 (The realizable exceptional component)**.** *On $\mathcal{M}_{AB}$, write*

$$y=z=a,\quad u=v=b,\quad x=a+b,\quad w=c.$$

The exact Euclidean domain is

$$a,b>0,\qquad 0<c<\frac{4ab}{a+b},\qquad V^2=\frac{c(4ab-(a+b)c)}{144}.$$

Within this domain $\mathcal{M}_{AB}$ is disjoint from $\mathcal{O}$ and $\mathcal{D}$, and meets the jaw locus exactly when $a=b$. Distinct distinguished-edge components are disjoint.

Proof. Place AB on a coordinate axis with length $\sqrt{a+b}$. The vertices C,D have the same axial coordinate and lie on a circle of radius $\sqrt{ab/(a+b)}$. Their squared chord length is c. Non-coplanarity is equivalent to the two strict chord bounds displayed above; the scalar triple product gives the volume formula. The orthocentric equations force $c=0$. The equifacial equations force $a=b$ and $c=2a$, the other degenerate endpoint. Inspecting the three equal-cycle patterns shows that a jaw requires and is implied by $a=b$. Finally, two circumdiameters would give two antipodal vertex pairs, all contained in a plane through the circumcenter. □

For the projection locus, retain the earlier exclusion of equilateral faces; in this notation those occur when $c=a$ or $c=b$.

There are six labeled diametral-mirror components, one for each distinguished edge, but vertex relabeling makes them one geometric family. There are similarly three labeled jaw components, corresponding to the three Hamiltonian four-cycles in the tetrahedral edge graph.

Modulo similarity, every dimension in Table 1 drops by one: orthocentric shapes form a three-dimensional family, whereas jaw, equifacial, and diametral-mirror shapes each form two-dimensional families.

11 Concluding remarks and open directions

The distinctive central-projection mechanism is the circumcenter lying in two adjacent face planes. Their common edge is a circumdiameter, and the remaining equations select diametral mirrors. The orthocentric, equifacial, and jaw families are classical; the results claimed here are their exact roles in the two all-face classifications and the exceptional central component. The metric and symmetry background is documented in [5, 11].

Several questions suggest themselves.

1. What classifications arise for oblique affine projections, or for central projections whose center is not the opposite vertex?

2. Which other tetrahedral center lines have analogous all-face orthogonal and cevian projection classifications?
3. Can one characterize higher-dimensional simplices whose Euler line projects onto the Euler line of every facet or triangular 2-face, under either projection mechanism?
4. What are the singularities and intersection multiplicities of the projective closures of the OEPP and CEPP edge varieties?
5. Do jaws or diametral mirrors have distinguished extremal properties involving dihedral angles, inradius, circumradius, or normal-frame isotropy?

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Proof transparency. All classifications in this article follow from the displayed projection identities, the six-distance determinant, and the elementary factorization of the four edge equations. No computer-algebra calculation or external Mathematica notebook is used in a proof, and optional independent exact checks are included in the bundle.

A Geometric background: jaws as lifted rhombi

The word "jaw" and its construction from a rhombus originate in [5]. We record the following equivalent complete-diagonal translation model, with a proof, because it also makes the half-turn symmetry used here transparent.

Proposition A.1 (Lifted-rhombus model). *A tetrahedron is a jaw if and only if it can be obtained as follows: start with a planar rhombus and translate one of its complete diagonals rigidly by a nonzero vector perpendicular to the plane.*

Proof. Let AB_0CD_0 be a planar rhombus, and let $\mathbf{n}$ be normal to its plane. Put

$$B = B_0 + h\mathbf{n}, \qquad D = D_0 + h\mathbf{n}, \qquad h \neq 0.$$

The four original rhombus sides have equal length, and each new cycle edge acquires the same additional squared normal component h^2. Therefore

$$AB = BC = CD = DA,$$

and the resulting tetrahedron is a jaw.

Conversely, suppose $AB = BC = CD = DA$. Let

$$K = \frac{A+C}{2}, \qquad L = \frac{B+D}{2}$$

be the midpoints of the diagonals. Since both B and D are equidistant from A and C, the line BD lies in the perpendicular-bisector plane of AC; hence $BD \perp AC$ and $KL \perp AC$. Similarly, since A, C are equidistant from B, D, $KL \perp BD$. Thus KL is the common perpendicular of the two skew diagonals.

Translate the entire segment BD parallel to KL until L coincides with K. The translated diagonals are perpendicular and bisect one another, so their endpoints form a planar rhombus. Reversing this translation recovers the original jaw. □

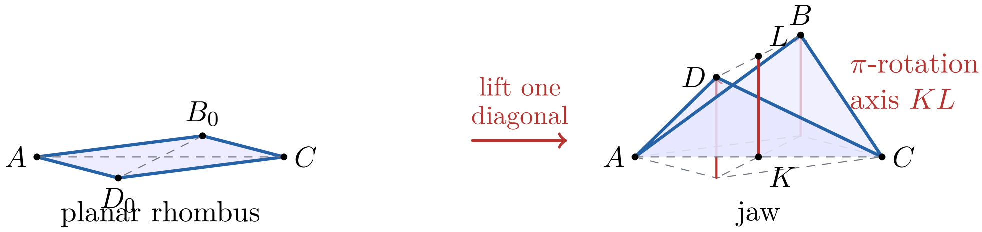

Figure 3: A jaw is a rhombus with one complete diagonal lifted normally. The line joining the diagonal midpoints is a half-turn symmetry axis.

Proposition A.2 (Symmetry of a jaw). *Every jaw has a half-turn symmetry about the line KL joining the midpoints of its diagonals. Its circumcenter, centroid, and Monge point all lie on this axis; consequently its tetrahedral Euler object is contained in the symmetry axis.*

Proof. By Proposition A.1, the two diagonals are perpendicular to one another and to KL. Rotation through π about KL sends $A \leftrightarrow C$ and $B \leftrightarrow D$, so it preserves the tetrahedron. Every uniquely defined center preserved by this rotation lies on its fixed axis. This applies to O, G, M. □

There is also an immediate facewise explanation of the projection property. Each face of a jaw is isosceles. For example, if $AB = BC = CD = DA$, then ABC is isosceles with base AC, while the opposite vertex D is equidistant from A, C. Its projection onto ABC therefore lies on the perpendicular bisector of AC, which is the Euler line when that isosceles face is non-equilateral. The same argument applies cyclically to all four faces.

Computational materials

The accompanying bundle contains editable sources, build instructions, independent exact-check scripts and their recorded outputs. These support the displayed calculations; open global classification problems are not certified by local checks.

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