# Dimensions of Tetrahedral $Z$-Cevian Families

Face constraints, jaws, tripods, and odd-dimensional obstructions

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**Abstract**

We study the dimensions of tetrahedral families obtained by imposing triangle conditions on all four faces or by requiring four cevians through prescribed facial centers to concur. A conormal-support argument proves that conic subanalytic facial sets of dimension at most two have a four-face lift of dimension at most three, including scale. This contains the case of a nonzero homogeneous analytic face equation and does not require symmetry. A separate argument using the omitted opposite edge in concurrence residuals excludes five-dimensional components for proper analytic concurrence families with nonvanishing facial weights. An explicit positive polynomial triangle-center function has a three-dimensional jaw component, certified by an exact Jacobian minor; its entire concurrence locus has dimension either three or four. We also classify all-isosceles tetrahedra: the maximal components are jaws and tripods, with six additional two-dimensional alternating-path components. Four further facial constraints have dimension two, supported by exact finite certificates, while positive gearing gives sharp dimension-three examples. The distinction between a guaranteed facial subset and the full concurrence locus is maintained throughout. Conditional generic constructions and catalogue questions are separated from the proved classifications.

**Keywords.** Triangle center; tetrahedron; cevian concurrence; face constraint; dimension; Gröbner basis.

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## 1 Introduction

**Notation across the series.** Throughout the series, $e_{ij} = |V_iV_j|$ denotes an edge length and $d_{ij} = e_{ij}^2$ its square. Dimensions include scale; passing to similarity classes subtracts one. Local six-variable aliases are specified at each metric calculation.

A triangle center is usually studied inside one triangle. The present paper uses it instead as a rule imposed simultaneously on the four faces of a tetrahedron. Given a triangle center $Z$, one joins each tetrahedral vertex to the copy of $Z$ in the opposite face. The resulting four lines are the $Z$-cevians. Their concurrence is a genuinely spatial compatibility condition: the three barycentric weights visible in each face must be restrictions of one set of four global weights.

This problem continues the program initiated in Hajja–Hammoudeh–Hayajneh–Martini [17]. That paper asked, in particular, what odd dimensions can occur for tetrahedral $Z$-cevian families. It also showed that every such family contains the two-dimensional kite locus, so dimension one is unavailable in the usual global sense. The present paper settles the five-dimensional part for real-analytic center rules with nonvanishing facial weights and exhibits genuine three-dimensional analytic components. We deliberately distinguish this local-component statement from the stronger assertion that a particular explicit center has no remote four-dimensional component; the latter is a global elimination problem.

A companion article introduces the four-dimensional power–$k$ families and their parallel Cevian–pedal theory (companion P). Two further companion papers study facial projection rather than concurrence: the tetrahedral Euler object (companion E) and the skeletal-centroid line (companion S). Thus the present article occupies the dimension-theoretic part of a broader study of how planar triangle centers constrain tetrahedral geometry.

A facial constraint $g = 0$ produces a guaranteed concurrence subset whenever the center reduces to the centroid on that constraint. Theorem 3.2 bounds the dimension of this subset by three, in the more general setting of conic subanalytic facial sets. The bound is sharp for the all-isosceles and geared families. Additional concurrence components require separate analysis.

Triangle centers are understood in the functional sense introduced by Kimberling [14, 15]; the online Encyclopedia of Triangle Centers (ETC) provides the standard catalogue [5, 16]. Our algebraic computations use standard Gröbner-basis and dimension methods [3, 6]. Since positivity, triangle inequalities, and the Cayley–Menger inequality are semialgebraic open conditions, they select physically realizable parts of algebraic components without increasing their dimension; see [2].

**Companion manuscripts.** The unpublished companion papers by the present author are designated (P) *Power–k tetrahedra: a unified Cevian–pedal theory of four-parameter tetrahedral families*; (E) *Tetrahedra with orthogonal and cevian Euler–projection properties: edge-length classifications with orthocentric, jaw, equifacial, and diametral-mirror components*; (S) *Skeletal-centroid projection properties of tetrahedra: circumscriptible families, jaws, and a symmetry principle.* All are version 2, dated 6 September 2026; these are local version identifiers, not public repository records. Results needed for the present proofs are stated and proved here.

## 2 Definitions and the basic inclusion

### 2.1 The edge cone

Label the edges of a tetrahedron $T = V_1V_2V_3V_4$ by

$$x_{ij} = |V_iV_j|, \qquad 1 \le i < j \le 4.$$

Let $\mathcal{E} \subset \mathbb{R}^6_{>0}$ be the set of edge vectors for which every facial triple satisfies the strict triangle inequalities and the Cayley–Menger determinant is positive. Thus $\mathcal{E}$ is the open semialgebraic cone of nondegenerate Euclidean tetrahedra, modulo rigid motion but not modulo scale. Every dimension in this paper includes scale. Passing to similarity classes subtracts one.

The four faces, written only in terms of their edges, are

$$\begin{aligned} F_1 &= (x_{23}, x_{24}, x_{34}), \quad F_2 = (x_{13}, x_{14}, x_{34}), \\ F_3 &= (x_{12}, x_{14}, x_{24}), \quad F_4 = (x_{12}, x_{13}, x_{23}). \end{aligned} \tag{1}$$

**Definition 2.1** (Analytic facial constraint)**.** An *analytic facial constraint* is a nonzero real-analytic function $g$ on the positive triangle cone such that

$$g(a, b, c) = g(a, c, b) = g(b, a, c), \qquad g(ta, tb, tc) = g(a, b, c)$$

for $t > 0$, and $g(1, 1, 1) = 0$. Its face-constrained tetrahedral locus is

$$\mathcal{S}_g = \{x \in \mathcal{E} : g(F_1) = g(F_2) = g(F_3) = g(F_4) = 0\}.$$

The symmetry requirement makes the equation independent of the labeling of a face. Degree zero means that it is a condition on the shape of a triangle. A rational homogeneous expression is analytic on the positive cone whenever its denominator has no zero there.

### 2.2 Triangle-center functions and $Z$-cevians

Let $F(a, b, c)$ be a homogeneous, bisymmetric barycentric center function in Kimberling's sense:

$$F(a, b, c) = F(a, c, b), \qquad F(ta, tb, tc) = t^d F(a, b, c).$$

The corresponding center of a triangle with side lengths $a, b, c$ has barycentrics

$$Z_F = (F(a, b, c) : F(b, c, a) : F(c, a, b)).$$

**Definition 2.2** ($Z$-cevian family)**.** For each $i$, place $Z_F$ in the face $F_i$ opposite $V_i$. The line joining $V_i$ to this facial point is the $i$-th $Z$-cevian. The family $\mathrm{ZCevian}[Z_F]$ consists of the tetrahedra in $\mathcal{E}$ for which all four $Z$-cevians concur.

For a constraint $g$, a positive constant $C$, and a real parameter $\lambda$, define

$$F_a^{g,\lambda}(a, b, c) = C + \lambda \frac{a}{a + b + c} g(a, b, c). \tag{2}$$

The other two weights are obtained cyclically. Because $g$ is symmetric and homogeneous of degree zero, this is a homogeneous bisymmetric center function. If positivity of all weights is desired, one may restrict $\lambda$ or the normalized shape domain. Projective barycentrics themselves do not require positivity.

**Proposition 2.3** (Guaranteed median component)**.** *For every analytic facial constraint $g$ and every $\lambda$,*

$$\mathcal{S}_g \subseteq \mathrm{ZCevian}[X_{g,\lambda}],$$

*where $X_{g,\lambda}$ is defined by (2).*

*Proof.* If a face satisfies $g = 0$, its three weights in (2) are all equal to $C$. The facial center is therefore its centroid. If all four faces satisfy $g = 0$, the four $X_{g,\lambda}$-cevians are the four tetrahedral medians, which concur at

$$G_T = \frac{V_1 + V_2 + V_3 + V_4}{4}.$$

□

*Remark* 2.4 (Dimension of the guaranteed subset)*.* The set of triangles satisfying $g = 0$ is not the object whose dimension should be compared directly with a tetrahedral $Z$-cevian family. The appropriate object is its *four-face lift* $\mathcal{S}_g$. Proposition 2.3 gives

$$\dim \mathrm{ZCevian}[X_{g,\lambda}] \geq \dim \mathcal{S}_g.$$

Equality requires an additional assertion excluding maximal-dimensional accidental-concurrence components.

## 3 Two general dimension theorems

### 3.1 A sharp upper bound for $\mathcal{S}_g$

We first isolate the elementary linear-algebra fact behind the bound. The six coordinates of $\mathbb{R}^6$ are indexed by the edges of $K_4$; the four facial triples are its four triangles.

**Lemma 3.1** (Four triangular supports)**.** *For $i = 1, \ldots, 4$, let $r_i \in \mathbb{R}^6$ be nonzero and supported on the three edges of the face $F_i$. If every $r_i$ has at least two nonzero coordinates, then*

$$\mathrm{rank}(r_1, r_2, r_3, r_4) \geq 3.$$

*Proof.* Any two facial supports meet in exactly one edge. Since each vector has at least two nonzero coordinates, no two vectors belonging to different faces can be proportional. Suppose, for contradiction, that all four vectors span a plane. Use $r_1, r_2$ as a basis.

The support of $r_1$ is $\{23, 24, 34\}$, and that of $r_2$ is $\{13, 14, 34\}$. To form a vector supported on $F_3 = \{12, 14, 24\}$, a linear combination of $r_1, r_2$ must eliminate the private coordinates 23 and 13, as well as the shared coordinate 34. A nonzero combination can do this only in the exceptional support pattern

$$\operatorname{supp} r_1 = \{24, 34\}, \qquad \operatorname{supp} r_2 = \{14, 34\},$$

up to interchanging the two faces and their private edges. In that case the resulting vector is supported on $\{14, 24\}$. But the entire span is then supported on $\{14, 24, 34\}$, which is disjoint from $F_4 = \{12, 13, 23\}$. It cannot contain the nonzero vector $r_4$. The remaining support cases force one coefficient to vanish and are even more immediate. Hence the rank is at least three. $\square$

**Theorem 3.2** (Conic facial constraints)**.** *For each face let $A_i$ be a subanalytic, scale-invariant subset of the positive triangle cone, of dimension at most two. The realizable six-edge locus whose four facial triples belong to their respective $A_i$ has dimension at most three. In particular,* $\dim \mathcal{S}_g \leq 3$ *for every nonzero homogeneous real-analytic facial constraint $g$. Symmetry of $g$ and vanishing at the equilateral triangle are not needed for this upper bound.*

*Proof.* Intersect each $A_i$ with the perimeter-one triangle domain

$$\Delta = \{(a, b, c) > 0 : a + b + c = 1,\ \max(a, b, c) < 1/2\}.$$

Take a locally finite subanalytic smooth stratification of these sets, then cone each stratum by positive scaling. The resulting facial strata have dimension at most two. Their tangent spaces contain the positive radial vector $(a, b, c)$. Consequently any nonzero conormal satisfies $a\nu_a + b\nu_b + c\nu_c = 0$, and has at least two nonzero entries.

The intersection of the four facial pullbacks is subanalytic and admits a smooth refinement compatible with the chosen facial strata [18, 2]. At any point choose one nonzero conormal from each face and pull it back to six-edge space. Each is supported on its facial triangle and has at least two nonzero entries. Lemma 3.1 gives rank at least three. The tangent space of each refined intersection stratum lies in their common kernel, so it has dimension at most $6 - 3 = 3$.

For the stated analytic consequence, a nonzero analytic function on the connected triangle cone has a zero set of dimension at most two. Homogeneity makes this zero set conic, so the preceding argument applies. $\square$

*Remark* 3.3. The dimension hypothesis on the facial sets is essential: a proper conic set can still contain an open subset and impose no local restriction. Several simultaneous homogeneous analytic equations are covered whenever their common zero set has dimension at most two.

## 3.2 The dimension-five obstruction on the nonvanishing-weight domain

Write $w_i^{ijk}$ for the barycentric weight assigned to $V_i$ by the center $Z$ in the face $V_iV_jV_k$. On the nonzero-weight domain, the four cevians concur projectively if and only if there exists a global weight vector, not identically zero, $(W_1, W_2, W_3, W_4)$ whose restrictions to every face are proportional to its three local weights. In particular, for the two faces $ijk$ and $ij\ell$ containing the edge $ij$, one must have

$$R_{ij} := w_i^{ijk} w_j^{ij\ell} - w_j^{ijk} w_i^{ij\ell} = 0. \tag{3}$$

The crucial observation is

$$R_{ij} \text{ is independent of } x_{k\ell},$$

the edge opposite $ij$. A projective concurrence point is finite exactly when $\sum_i W_i \neq 0$. The six equations $R_{ij} = 0$ are redundant, but their common zero set is exactly the compatibility locus wherever the weights are defined and nonzero.

**Lemma 3.4** (Cylinder lemma). *Let $U = U_0 \times I \subset \mathbb{R}^{n-1} \times \mathbb{R}$, and let $F(x_1, \dots, x_{n-1})$ be a nonzero real-analytic function that is independent of $x_n$. Every irreducible codimension-one analytic component of $Z(F) \cap U$ is locally a product $H_0 \times I$. Consequently, if an irreducible analytic hypersurface $H \subset U$ is contained in $Z(F)$, then near a generic point of $H$ it is cylindrical in the $x_n$-direction.*

*Proof.* One has $Z(F) \cap U = (Z(F) \cap U_0) \times I$. The maximal-dimensional irreducible components are therefore products of irreducible components in $U_0$ with $I$. If an irreducible hypersurface $H$ is contained in $Z(F)$, then it has the same local dimension as one of these maximal components and hence agrees with it on a nonempty relatively open subset. This gives the asserted local product structure at generic points. □

**Theorem 3.5** (The dimension-five obstruction). *Let $Z$ have a real-analytic barycentric center function $f(a, b, c)$ that is nowhere zero on the positive triangle cone. If* ZCevian$[Z]$ *is a proper subset of $\mathcal{E}$, then it has no five-dimensional analytic component. In particular,*

$$\dim \mathrm{ZCevian}[Z] \neq 5.$$

*The same conclusion holds on any connected relabeling-invariant open subdomain on which all cyclic facial weights are real analytic and nonzero and concurrence is interpreted projectively. For the global nowhere-zero real rule, all cyclic weights have the same sign, so the face centers and the concurrence point are automatically finite.*

*Proof.* The six residuals are carried into one another by relabeling the vertices. If one residual, say $R_{12}$, were identically zero, then all six would be identically zero. Because all facial weights are nonzero, the six vanishing pairwise cross-products make the local weight ratios on adjacent faces universally compatible. They therefore glue to a global weight vector for every tetrahedron, so the four $Z$-cevians would concur identically. This would give ZCevian$[Z] = \mathcal{E}$, contrary to properness. Hence none of the six residuals is identically zero.

Assume that $H$ is an irreducible five-dimensional analytic component of ZCevian$[Z]$. Then $H$ is locally a hypersurface in the six edge variables and is contained in every zero set $R_{ij} = 0$. Since $R_{12}$ is independent of $x_{34}$, Lemma 3.4 shows that, near a generic point of $H$, the hypersurface is cylindrical in the $x_{34}$-direction. The residual $R_{34}$ omits $x_{12}$, and the four remaining residuals similarly supply the other four coordinate directions. Because there are only six exceptional analytic subsets, one may choose a single smooth point of $H$ that is generic for all six applications. At that point every coordinate basis vector belongs to $TH$, forcing $\dim TH \geq 6$, contrary to $\dim H = 5$. □

*Remark* 3.6. The nonvanishing hypothesis is used only to turn identically vanishing edge residuals into universal compatibility. Center rules with persistent zero barycentric weights may have additional lower-dimensional singular strata and require a separate gluing argument; no claim about such zero-weight strata is made here.

*Remark* 3.7. The theorem excludes dimension five, not dimension four. Power–$k$ tetrahedra and the isogonic Eisenstein family provide important four-dimensional $Z$-cevian families (companion P). Nor does the theorem force even dimension: the next result gives an explicit analytic three-dimensional component.

## 3.3 An explicit three-dimensional analytic component

Define the homogeneous bisymmetric center function

$$f_*(a, b, c) = (a + b + c)^4 + \frac{1}{100}(a - b)(a - c)(b - c)^2. \tag{4}$$

It is symmetric in $b, c$, and therefore defines a real-analytic triangle center in Kimberling's functional sense. It is also strictly positive on the positive triangle cone: each of the four difference factors has absolute value less than $a+b+c$, so the perturbation has absolute value less than $(a+b+c)^4/100$.

**Theorem 3.8** (A genuine three-dimensional $Z$-cevian component)**.** *The $Z$-cevian locus associated with (4) has a genuine three-dimensional local component at the jaw with $(L,p,q)=(5,6,7)$; in a neighborhood of this point the concurrence locus equals the jaw manifold.*

*Proof.* On an isosceles triangle, each of the three cyclic perturbation terms in (4) vanishes. Thus the corresponding center is the centroid of every isosceles triangle. Every face of a jaw is isosceles, so all four $Z_{f_*}$-cevians are medians and concur at the tetrahedral centroid. Hence the three-parameter jaw manifold

$$x_{12}=x_{23}=x_{34}=x_{14}=L, \qquad x_{13}=p, \qquad x_{24}=q$$

is contained in the concurrence locus.

To certify the local dimension, form the six compatibility residuals (3) and differentiate with respect to $(x_{12},x_{13},x_{14},x_{23},x_{24},x_{34})$. At the admissible jaw $(L,p,q)=(5,6,7)$, the Cayley–Menger determinant gives

$$V^2=\frac{735}{4}>0,$$

and the residual vector vanishes exactly. The submatrix formed by residuals $R_{12}, R_{13}, R_{14}$ and columns $x_{12}, x_{14}, x_{23}$ is

$$J_0=\begin{pmatrix} 607809/100 & -131072/25 & -83521/100 \\ 32768/25 & -32768/25 & -32768/25 \\ 131072/25 & -607809/100 & 0 \end{pmatrix}, \qquad \det J_0=-\frac{179359981764608}{15625}\neq 0.$$

Thus the full residual Jacobian has rank at least three. On the other hand, the residual map vanishes identically on the smooth three-parameter jaw manifold, so its differential annihilates the three-dimensional jaw tangent space and has rank at most three. Consequently its rank is exactly three. Three independent residuals cut out a local manifold of dimension $6-3=3$ containing the jaw manifold; the full concurrence locus is contained in that manifold and already contains the jaw manifold. Hence the local component through this point is exactly three-dimensional. $\square$

**Corollary 3.9** (Global alternatives)**.** *The entire realizable concurrence locus of $f_*$ has dimension either three or four.*

*Proof.* Theorem 3.8 supplies the lower bound three and a nonzero residual derivative, so the analytic residuals are not identically zero. The realizable edge cone is connected (positive-definite Gram matrices give connected squared-edge coordinates). Analytic uniqueness therefore excludes an open six-dimensional concurrence set. Positivity of the weights allows Theorem 3.5 to exclude dimension five. The only remaining dimensions are three and four. $\square$

The unresolved step is to rule out remote four-dimensional components. The local minor, branch enumerations for the separate facial constraints, and a numerical search cannot supply that global certificate. This is the remaining distinction in the odd-dimensionality question of [17].

# 4 The isosceles constraint and the Jaws–Tripods theorem

Let $s=a+b+c$, and set

$$g_1(a,b,c)=\frac{(a-b)^2(b-c)^2(c-a)^2}{s^6}. \tag{5}$$

Then $g_1=0$ if and only if the triangle is isosceles, with equilateral triangles included.

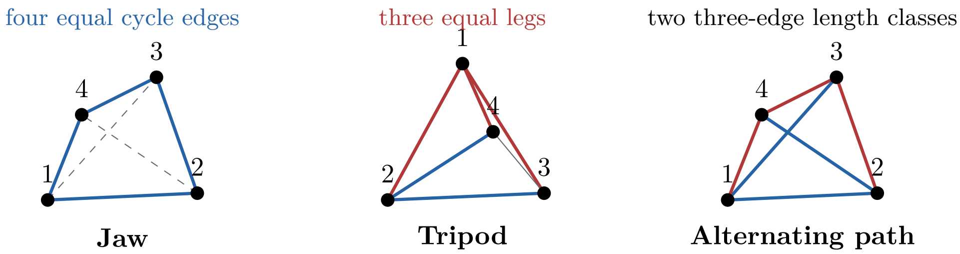


Figure 1: The maximal three-dimensional all-isosceles components are jaws and tripods. Six two-dimensional alternating-path components are also needed for the full set-theoretic classification.

**Definition 4.1** (Jaw)**.** A *jaw* is a tetrahedron whose vertices can be cyclically labeled so that four cycle edges are equal:

$$x_{12} = x_{23} = x_{34} = x_{14} = L,$$

while the two diagonals $x_{13} = p$ and $x_{24} = q$ are free, subject to the Euclidean inequalities.

**Definition 4.2** (Tripod)**.** A *tripod* is a tetrahedron with three equal edges incident at one vertex and with an isosceles opposite face. For example,

$$x_{12} = x_{13} = x_{14} = L, \qquad x_{23} = x_{24} = p, \qquad x_{34} = q.$$

There are four choices of the distinguished vertex and three choices of the equal pair in the opposite face.

**Definition 4.3** (Alternating-path stratum)**.** An *alternating-path* tetrahedron has, after relabeling,

$$x_{12} = x_{13} = x_{24} = u, \qquad x_{14} = x_{23} = x_{34} = v. \tag{6}$$

Each of the two length classes forms a three-edge path on the four vertices. There are six labeled components.

**Theorem 4.4** (Jaws–Tripods theorem, complete form)**.** *The all-isosceles tetrahedral locus $\mathcal{S}_{g_1}$ has dimension three. Its maximal three-dimensional linear components in edge-length space are exactly*

*(i) three labeled jaw components;*

*(ii) twelve labeled tripod components.*

*The irredundant set-theoretic decomposition additionally contains six two-dimensional alternating-path components of the form* (6)*. Thus the phrase "exactly the union of jaws and tripods" is correct for the maximal-dimensional part, but not for the full locus unless the six residual strata are mentioned.*

*Proof.* Every face must choose one of its three side equalities. Hence $\mathcal{S}_{g_1}$ is the union of $3^4 = 81$ linear branch assignments. A branch is represented by a $4 \times 6$ matrix with one row $x_e - x_f = 0$ for each face.

We first classify the rank-three branches without computation. At a fixed vertex, inspect its three incident edges.

If all three are equal, the opposite face must be isosceles; this is a tripod. If exactly two are equal, say $x_{12} = x_{13} \neq x_{14}$, the two faces containing $x_{14}$ force

$$x_{24} \in \{x_{12}, x_{14}\}, \qquad x_{34} \in \{x_{13}, x_{14}\}.$$

If both choices use the first length, four edges form an equal cycle, giving a jaw. If both use the second, one obtains a tripod centered at $V_4$. In the mixed cases the fourth face either forces a tripod at $V_2$ or $V_3$, or introduces a fourth independent equality, leaving a two-dimensional branch.

If the three incident edges are pairwise distinct, each opposite edge is forced to equal one of the two adjacent incident edges. The fourth face can be isosceles only when two of these choices agree. This produces a tripod centered at one of the other vertices, unless four independent equalities are created. Therefore every rank-three branch is a jaw or a tripod.

Conversely, each jaw and each tripod makes all four faces isosceles and is defined by three independent linear equalities. There are three Hamiltonian cycles in $K_4$, hence three labeled jaws. There are $4 \cdot 3 = 12$ labeled tripods.

For completeness, exact row reduction of all 81 assignments gives

| matrix rank | 3 | 4 |
|---|---|---|
| branch assignments | 15 | 66 |
| distinct linear spaces | 15 | 24. |

Eighteen of the distinct rank-four spaces lie inside one of the rank-three components. The remaining six are precisely the relabelings of (6). This proves the irredundant decomposition.

Finally, the listed components meet the nondegenerate Euclidean cone in relatively open sets. For example, a tripod with $(L, p, q) = (3, 4, 5)$ is realizable, with exact value

$$V^2 = \frac{2375}{144} > 0.$$

For an alternating-path component, $(u, v) = (1, 3/2)$ has

$$V^2 = \frac{143}{9216} > 0.$$

Hence the largest realized components really have dimension three. □

# 5 Four generic two-dimensional constraints

Let $s = a + b + c$, and define cyclically

$$Q_A = b^2 + c^2 - a^2 - bc, \tag{7}$$
$$M_A = 2a(b^2 + c^2 - a^2) - b(c^2 + a^2 - b^2) - c(a^2 + b^2 - c^2). \tag{8}$$

The following symmetric degree-zero functions encode four natural disjunctive constraints:

$$g_2 = \frac{Q_A Q_B Q_C}{s^6}, \tag{9}$$
$$g_3 = \frac{M_A M_B M_C}{s^9}, \tag{10}$$
$$g_4 = \frac{(2a - b - c)(2b - c - a)(2c - a - b)}{s^3}, \tag{11}$$
$$g_5 = \frac{(2a^2 - b^2 - c^2)(2b^2 - c^2 - a^2)(2c^2 - a^2 - b^2)}{s^6}. \tag{12}$$

Table 1: The four constraints and their exact positive-chart dimensions. The chart dimension is computed after setting $x_{34} = 1$. Because every point of $\mathcal{E}$ has $x_{34} > 0$, adjoining scale gives the positive-cone dimension.

| | Geometric meaning of one branch | branches | chart dim. | cone dim. |
|---|---|---|---|---|
| $g_2$ | one angle is $60°$ | $3^4 = 81$ | 1 | 2 |
| $g_3$ | one angle cosine is the mean of the other two | 81 | 1 | 2 |
| $g_4$ | one side is the arithmetic mean of the other two | 81 | 1 | 2 |
| $g_5$ | one squared side is the mean of the other two squares | 81 | 1 | 2 |

Indeed, $Q_A = 0$ is the cosine-law equation $\cos A = 1/2$. Multiplying $2\cos A - \cos B - \cos C = 0$ by $2abc$ gives $M_A = 0$. The meanings of $g_4$ and $g_5$ are immediate.

**Proposition 5.1.** *For each $i = 2, 3, 4, 5$, every one of the 81 branch systems has Krull dimension one in the chart $x_{34} = 1$. Consequently, every branch meeting $\mathcal{E}$ has dimension at most two after scale is restored. Moreover, $\mathcal{S}_{g_i} \cap \mathcal{E}$ contains a two-dimensional Euclidean subfamily. Hence*

$$\boxed{\dim \mathcal{S}_{g_i} = 2, \qquad i = 2, 3, 4, 5.}$$

*Exact computational proof.* Each face has three possible cyclic branches, so each $g_i$ gives 81 branch systems. We dehomogenized by $x_{34} = 1$ and computed the Krull dimension of every branch ideal over $\mathbb{Q}$, using a degree-reverse-lexicographic Gröbner basis. For each of $g_2, g_3, g_4, g_5$, all 81 ideals had dimension one in the chart. Because every Euclidean edge vector has $x_{34} > 0$, restoring scale gives the required upper bound two on the positive Euclidean locus. The summary is Table 1.

This is an exact symbolic calculation, not a numerical rank test. The short reproducing program is given in Appendix C.

For the lower bound, choose a nontrivial smooth one-parameter family of acute triangles on one branch of the corresponding constraint. Such families are immediate for $g_2, g_4, g_5$; for $g_3$, in angle coordinates, put

$$F(B, C) = \cos(\pi - B - C) - \frac{\cos B + \cos C}{2}.$$

At the equilateral point,

$$\left.\frac{\partial F}{\partial B}\right|_{B=C=\pi/3} = \frac{3\sqrt{3}}{4} \neq 0,$$

so the implicit-function theorem supplies the required acute family. Every acute triangle is the face of a disphenoid whose four faces are congruent to it: in the standard $(\pm x, \pm y, \pm z)$ realization one may take, cyclically,

$$x^2 = \frac{b^2 + c^2 - a^2}{8} > 0.$$

The shape parameter together with scale therefore gives a two-dimensional Euclidean subfamily of each $\mathcal{S}_{g_i}$. □

*Remark* 5.2. Some individual branch ideals can have no positive Euclidean point, and distinct branch assignments can describe the same component. Neither issue can increase dimension; the disphenoid construction above supplies the needed global Euclidean lower bound. The calculation makes no claim about algebraic components confined to the coordinate hyperplane $x_{34} = 0$; such components cannot meet $\mathcal{E}$ and are irrelevant to the tetrahedral locus studied here.

## 6 A sixth constraint: 2:1 gearing

Put

$$E = (a-b)^2 + (b-c)^2 + (c-a)^2$$

and

$$R_2 = (a-2b)(b-2a)\,(b-2c)(c-2b)\,(c-2a)(a-2c).$$

Define

$$g_6(a,b,c) = \frac{E\,R_2^2}{s^{14}}. \tag{13}$$

The function is symmetric, nonnegative, homogeneous of degree zero, and analytic on the positive triangle cone. Moreover,

$$g_6 = 0 \quad \Longleftrightarrow \quad \text{the triangle is equilateral, or one side is twice another.}$$

Unlike $g_1$, it does not vanish on a generic isosceles triangle.

**Definition 6.1** (2-geared jaw)**.** A 2-geared jaw is a tetrahedron with a cyclic labeling such that

$$x_{12} = x_{34} = L, \qquad x_{23} = x_{14} = 2L, \qquad x_{13} = p, \quad x_{24} = q. \tag{14}$$

Every face of (14) contains the pair $L, 2L$, so every face satisfies $g_6 = 0$.

**Theorem 6.2** (Exact dimension of the $g_6$-locus)**.** *The face-constrained family $\mathcal{S}_{g_6}$ has dimension three. It contains the 2-geared jaws (14) as a natural three-dimensional subfamily.*

*Proof.* The function $g_6$ is a nonzero symmetric homogeneous real-analytic constraint on the positive triangle cone. Theorem 3.2 therefore gives

$$\dim \mathcal{S}_{g_6} \leq 3.$$

The geared-jaw equations (14) have three independent parameters $L, p, q$, so they supply the matching lower bound whenever they meet the nondegenerate Euclidean cone. They do: taking

$$L = 1, \qquad p = q = 2$$

gives

$$(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = (1, 2, 2, 2, 2, 1),$$

whose Cayley–Menger evaluation is

$$V^2 = \frac{7}{72} > 0.$$

Hence $\dim \mathcal{S}_{g_6} = 3$. No enumeration of the $7^4$ facial branch assignments is required. □

*Remark* 6.3 (A ratio parameter). For $\rho > 0$, one may replace 2 by $\rho$:

$$R_\rho = \prod_{\{x,y\} \subset \{a,b,c\}} (x - \rho y)(y - \rho x), \qquad g_\rho = \frac{E\,R_\rho^2}{s^{14}}.$$

The cyclic pattern $L, \rho L, L, \rho L$, with two free diagonals, has a nonempty Euclidean part for every $\rho > 0$. Indeed, choose both diagonals equal to $\max(1,\rho)L$. This is a disphenoid whose face has side lengths $L, \rho L, \max(1,\rho)L$; the face is strictly acute, so the disphenoid is nondegenerate. Realizability is open in the three pattern parameters. Theorem 3.2 supplies the matching upper bound, proving dimension exactly three for every fixed $\rho > 0$. Special values may merge facial branches, but they cannot raise the dimension.

# 7 Algorithms and reproducibility

The calculations in this paper use two complementary methods.

**Finite branch enumeration.** When $g$ is a product of explicitly listed factors, each face chooses one factor. A four-face system is obtained by taking the Cartesian fourth power of that branch list. Linear branches are settled by exact rational row reduction. For the Jaws–Tripods theorem, the human classification proves the maximal components and a finite check retains only the six irredundant alternating-path strata. The $g_6$ theorem needs no branch enumeration: Theorem 3.2 and the explicit geared jaws already give matching upper and lower bounds.

**Gröbner dimension.** For nonlinear branches, set one edge equal to 1, form the ideal over $\mathbb{Q}$, compute a Gröbner basis, and read the Krull dimension from the leading-term ideal or Hilbert polynomial. Since $\mathcal{E}$ lies in the positive chart $x_{34} > 0$, restoring scale adds one and proves the Euclidean upper bound in Proposition 5.1.

**Z-cevian dimension.** Given an algebraic center function $F$, form the six residuals (3), clear denominators, saturate by the denominators and by the Cayley–Menger boundary when appropriate, and compute the dimension of the resulting ideal. A primary decomposition distinguishes the guaranteed component $\mathcal{S}_g$ from remote accidental components. A Jacobian rank at one point certifies only a local lower-dimensional branch; it does not exclude a separate larger component.

**Minimal supplement.** No Mathematica notebook is required. The journal supplement consists of one short Sage script, `FaceConstraintChecks.sage`. It performs the finite rational row reductions for the all-isosceles branches, verifies the six alternating-path row spaces explicitly, and carries out the 81 exact Gröbner-dimension checks for each of $g_2, \dots, g_5$. The advertised counts are enforced by exact assertions. The dimension bounds, the $g_6$ theorem, the dimension-five obstruction, and the odd-dimensional local certificate are all proved directly in the text.

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# 8 Further questions

The exact local jaw certificate proves an odd-dimensional component. It does not determine the dimension of the entire concurrence locus of (4). The first remaining problem is to exclude remote four-dimensional components by a global saturated certificate. Further problems are the combinatorial transitions in the gearing ratio and a higher-simplex analogue of Lemma 3.1. Conditional generic mechanisms and catalogue comparisons are recorded in the appendices.

# A From $\mathcal{S}_g$ to $Z$-cevian families

## A.1 Guaranteed and additional concurrence

For the canonical center $X_{g,\lambda}$, the condition $g = 0$ turns every facial center into the centroid. Proposition 2.3 therefore gives a contained three-dimensional subset whenever $\dim \mathcal{S}_g = 3$. It does not establish that this subset is a component, or give equality of the two loci.

To see the source of the loophole, let $g_i = g(F_i)$. The edge residual (3) compares rational expressions in the weights on two adjacent faces. It can vanish because $g_i = g_j = 0$, but it can also vanish because two nontrivial ratios happen to agree. Four such agreements may define a component not contained in $\mathcal{S}_g$. Therefore one must distinguish:

$$\boxed{\mathcal{S}_g \subseteq \mathrm{ZCevian}[X_{g,\lambda}]} \qquad \text{from} \qquad \boxed{\mathcal{S}_g = \mathrm{ZCevian}[X_{g,\lambda}]}.$$

The first statement is automatic; the second requires elimination, primary decomposition, or an equivalent global certificate.

For $g_1$ and $g_6$, we can conclude unconditionally that

$$\dim \mathrm{ZCevian}[X_{g_i,\lambda}] \geq 3 \qquad (\lambda \neq 0).$$

On any connected relabeling-invariant regular domain where the facial weights are nonzero, Theorem 3.5 excludes dimension five. A proper family may nevertheless have dimension three or four. The centroid $\lambda = 0$ is universal and has dimension six.

### A.2 What an infinite set of centers does and does not prove

There are infinitely many center functions that collapse to the centroid on $g = 0$:

$$F_a = C + h(a,b,c)\, g(a,b,c), \tag{15}$$

where the cyclically generated multiplier is analytic and the resulting center function is homogeneous and bisymmetric. Positivity of $h$ may be imposed when it is geometrically desirable; it is not needed for Proposition 2.3.

Cardinality alone does *not* prove that one member has $Z$-cevian dimension exactly three. It is logically possible, in the absence of further information, that every member of a one-parameter family possesses a four-dimensional accidental component. Within the nonvanishing-weight scope of Theorem 3.5, the impossibility of dimension five does not rule this out.

The correct generic statement uses transversality.

**Proposition A.1** (Conditional generic transversality bound)**.** *Let $\mathcal{H}$ be a finite-dimensional real-analytic parameter manifold of equivariant multipliers in* (15)*. For each parameter $h$, form the four necessary edge residuals*

$$\mathcal{R}_h = (R_{12}^{(h)}, R_{13}^{(h)}, R_{14}^{(h)}, R_{23}^{(h)}),$$

*and let*

$$\mathcal{R} : (\mathcal{E} \setminus \mathcal{S}_g) \times \mathcal{H} \longrightarrow \mathbb{R}^4, \qquad (x,h) \longmapsto \mathcal{R}_h(x).$$

*Suppose that $\mathcal{R}$ is transverse to $0$. Then for a residual set of $h \in \mathcal{H}$, the accidental-concurrence locus has dimension at most two. Consequently, if $\dim \mathcal{S}_g = 3$, then*

$$\dim \mathrm{ZCevian}[X_{g,h}] = 3$$

*for generic $h$, provided the family is evaluated on the regular domain of its barycentric weights.*

*Proof.* Every accidental-concurrence point satisfies all six edge residuals, and hence in particular

$$(\mathrm{ZCevian}[X_{g,h}] \setminus \mathcal{S}_g) \subseteq \mathcal{R}_h^{-1}(0).$$

No sufficiency claim for the chosen four residuals is needed. By the parametric transversality theorem [7, 12], a generic partial map $\mathcal{R}_h : \mathcal{E} \setminus \mathcal{S}_g \to \mathbb{R}^4$ is transverse to 0. Its zero set is therefore either empty or a smooth set of dimension $6 - 4 = 2$. The accidental-concurrence subset has dimension at most two, while the guaranteed subset $\mathcal{S}_g$ has dimension three and dominates the union. $\square$

*Remark* A.2. Proposition A.1 is conditional: its conclusion applies only after the stated transversality hypothesis for the chosen parameter family has been verified. Under that hypothesis it is an existence mechanism, but a particular aesthetically chosen instance, such as $h = a/s$, still needs its own global certificate. This is the precise place where a Gröbner-basis or Hilbert-polynomial computation enters.

# B Canonical constraint centers and catalogue status

For $C = 1, \lambda = 1$, equivalently $h = a/s$ in (15), all six proposed centers have barycentrics

$$X_{g_i} = \left(1 + \frac{a}{s} g_i : 1 + \frac{b}{s} g_i : 1 + \frac{c}{s} g_i\right). \tag{16}$$

Because $g_i$ is fully symmetric, these points lie on the incenter–centroid line $X(1)X(2)$: their barycentric vectors are affine combinations of $(a, b, c)$ and $(1, 1, 1)$, with a triangle-dependent coefficient.

The normalization is part of the definition. Replacing $g_i$ by a nonzero scalar multiple leaves its zero set unchanged but usually produces a different triangle center in (16). Thus one face constraint gives a continuum of centers, not one canonical ETC entry.

No ETC identification is asserted here. A finite failed search would not prove nonoccurrence, because catalogues evolve and projectively equivalent formulas may look very different. The six explicit functions (16) are instead proposed as natural candidates for future exact comparison, with $g_1$ and $g_6$ carrying the strongest tetrahedral significance. Any identification should be certified by clearing denominators and proving cyclic projective proportionality symbolically; catalogue recognition is not used in any theorem of this paper.

# C Exact finite certificate

The accompanying file `FaceConstraintChecks.sage` is the finite-branch certificate. The additional `independent_checks.py` reproduces the odd-center Jacobian minor, volume checks, and component identities. It uses exact rational arithmetic. For $g_1$ it enumerates the $3^4 = 81$ choices of one equality per face, row-reduces their coefficient matrices, removes duplicate row spaces, and tests containment. Its output is

| rank | 3 | 4 |
|---|---|---|
| assignments | 15 | 66 |
| distinct row spaces | 15 | 24, |

with 18 of the rank-four spaces contained in a rank-three space and 6 irredundant rank-four spaces. The script additionally generates the six vertex-relabelings of the alternating-path row space, asserts exact set equality with those six irredundant spaces, and prints `g1 alternating-path spaces verified 6`. Thus the finite certificate checks the geometric identification used in the body of the paper.

For $g_2, \ldots, g_5$, the same file sets $x_{34} = 1$, forms all 81 branch ideals for each constraint over $\mathbb{Q}$, and computes their Krull dimensions. The exact output is

```
g2 {1: 81}
g3 {1: 81}
g4 {1: 81}
g5 {1: 81}
```

Restoring the scale coordinate gives positive-cone dimension two for every branch meeting $\mathcal{E}$. No statement is made about components confined to $x_{34} = 0$. The script contains the factor definitions themselves; no omitted helper, external ETC table, random sampling, floating-point rank, or Mathematica notebook is needed.

## Computational materials

The accompanying bundle contains editable sources, build instructions, independent exact-check scripts and their recorded outputs. These support the displayed calculations; open global classification problems are not certified by local checks.

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