Power-k Tetrahedra: A Unified Cevian–Pedal Theory of Four-Parameter Tetrahedral Families

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Abstract

We develop a common construction of tetrahedral Cevian and pedal concurrence from symmetric analytic edge laws that are homogeneous and strictly increasing in positive vertex parameters. Each such law gives a four-dimensional family and two facial center rules whose concurrence loci recover that family exactly. We then study the power law $e_{ij}^k = p_i + p_j$, including signed parameters. On nonzero sign chambers the Cevian characterization is projective; the extra conditions for finite concurrence are stated explicitly. The pedal converse holds on every such chamber for positive exponents and on the positive branch for every nonzero exponent. The classical circumscriptible and orthocentric families occur at exponents one and two, while a normalized limit yields the isodynamic family. An Eisenstein-power construction supplies another explicit family. Its exponent-one member has the first Fermat point as both facial centers, although the two spatial concurrence points are generally distinct; the pedal point is the incenter. Exact barycentric formulas, residual tests, and reproducible vector examples connect these constructions. Global algebraic certification is distinguished from local Jacobian dimension tests, and the mixed-sign negative-exponent pedal converse remains an open problem.

Keywords. Triangle center; tetrahedron; Cevian concurrence; pedal concurrence; homogeneous edge law; Fermat point.
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1 Introduction

Notation across the series. Throughout the series, $e_{ij} = |V_iV_j|$ denotes an edge length and $d_{ij} = e_{ij}^2$ its square. Dimensions include scale; passing to similarity classes subtracts one. Local six-variable aliases are specified at each metric calculation.

The passage from a planar center to four corresponding face centers is one of the most productive ways to connect triangle geometry with tetrahedral geometry. Given a tetrahedron, one may join each vertex to a prescribed center of the opposite face, or erect a normal to each face at that center. Concurrence of the first four lines is an affine–barycentric condition; concurrence of the second four is a Euclidean metric condition. Classical examples include the Gergonne cevians of a circumscriptible tetrahedron, the orthocenter cevians and normals of an orthocentric tetrahedron, and the Fermat cevians of an isogonic tetrahedron [1, 14, 8].

The purpose of this paper is to organize many such phenomena by a single exponent. The elementary-looking law

$$|V_iV_j|^k = p_i + p_j$$

turns out to carry two parallel theories. The parameters p_i are allowed to be real; Euclidean edge lengths require every pair sum $p_i + p_j$ to be positive. On every triangular face the three additive parameters are reconstructed directly from the side lengths by the power-law cosine defects

$$q_{k,a} = \frac{b^k + c^k - a^k}{2}, \qquad \text{cyclically.}$$

Signed powers of these defects give barycentric Cevian weights. On the positive branch, the quantities $p_i^{1/k}$ may also be interpreted as radii, and their squares give the radical-center construction underlying the pedal theory. For $k > 0$ this pedal construction continues through the signed chambers by using real signed powers.

This parametrization is deliberately broader than the positive one. At $k = 1$ Euclidean triangle inequalities force positivity anyway, whereas at $k = 2$ allowing signed p_i is exactly what recovers the full orthocentric family. For negative k, the Cevian theory remains valid throughout the nonzero-defect domain, but the global pedal converse across mixed-sign components is more delicate because inverse signed powers have singular walls; the positive branch remains exact.

Several further cautions are important. At $k = 0$ the displayed law has no literal value, and the continuation is taken on the normalized positive branch through the power mean. The signed power $x^{\langle t\rangle}$ is real analytic separately on $x > 0$ and $x < 0$, but not across $x = 0$ in general; accordingly, every statement involving a nonintegral or negative power is made on a connected open sign chamber. Zero defects form lower-dimensional boundary strata and are included only when an explicit continuous or projective extension is stated. The residual formalism developed below is also used in the companion study of lower-dimensional and face-constrained Z-Cevian families (companion D).

Companion manuscripts. The unpublished companion papers by the present author are designated (D) *Dimensions of Tetrahedral Z-Cevian Families: Face constraints, jaws, tripods, and odd-dimensional obstructions*; (E) *Tetrahedra with Orthogonal and Cevian Euler–Projection Properties: Edge-length classifications with orthocentric, jaw, equifacial, and diametral-mirror components*; (S) *Skeletal-Centroid Projection Properties of Tetrahedra: Circumscriptible families, jaws, and a symmetry principle.* All are version 2, dated 6 September 2026; these are local version identifiers, not public repository records. Results needed for the present proofs are stated and proved here.

2 Triangle centers and tetrahedral concurrence

Let ABC be a triangle with side lengths $a = |BC|$, $b = |CA|$, $c = |AB|$. A barycentric center function is a homogeneous function $f(a, b, c)$ satisfying the bisymmetry

$$f(a, b, c) = f(a, c, b).$$

It defines the triangle center

$$Z_f(ABC) = \big(f(a, b, c) : f(b, c, a) : f(c, a, b)\big). \tag{1}$$

Unless stated otherwise, the functions used here are real analytic on each open admissible chamber. Barycentric coordinates are homogeneous. For an affine combination, we first normalize each triple to have coordinate sum one.

Let $T = V_1V_2V_3V_4$ be a nondegenerate tetrahedron and let F_i be the face opposite V_i. Denote by Z_i the copy of Z in F_i.

Definition 2.1 (Z-Cevian and Z-pedal families)**.** The family ZCevian$[Z]$ consists of tetrahedra for which the four lines V_iZ_i concur in real projective three-space. Zero-sum barycentric triples represent points at infinity; the line from a finite vertex to such a point has its usual parallel-direction interpretation. Write ZCevian$^{\text{fin}}[Z]$ when all facial centers and the concurrence point are required to be finite. The family ZPedal$[Z]$ consists of tetrahedra for which the four lines through Z_i perpendicular to F_i concur.

The universal pedal center is the circumcenter $X(3)$: the normal at the circumcenter of every face passes through the tetrahedral circumcenter. Likewise, the centroid $X(2)$ is the universal

Cevian center by Commandino's theorem. Thus these two centers must be excluded when a concurrence family is meant to select a proper four-parameter subfamily of the six-parameter space of tetrahedra.

Lemma 2.2 (Global-weight criterion)**.** *Assume every facial barycentric coordinate is nonzero. The four Z-cevians concur projectively if and only if there are nonzero global weights $W_1, \ldots, W_4$ such that, on every face, the barycentric coordinates of its Z-center are proportional to the three corresponding W_j.*

Proof. If $Q = (W_1 : W_2 : W_3 : W_4)$ in tetrahedral barycentric coordinates, then the intersection of V_iQ with F_i has the three remaining coordinates $(W_j)_{j\neq i}$. The concurrence point is finite precisely when $\sum_j W_j \neq 0$, and the center on F_i is finite precisely when $\sum_{j\neq i} W_j \neq 0$. The converse follows by extending any face triple by a zero in the omitted coordinate and comparing the four concurrent lines. □

Lemma 2.3 (Radical-center criterion)**.** *Fix real power weights ρ_A, ρ_B, ρ_C on a triangle ABC. When they are nonnegative they may be viewed as squared radii; algebraically they may have either sign. The radical center of the three vertex-centered power circles has barycentrics $(g_a : g_b : g_c)$, where*

$$L_a = b^2 + c^2 - a^2, \qquad L_b = c^2 + a^2 - b^2, \qquad L_c = a^2 + b^2 - c^2, \tag{2}$$

$$g_a = a^2 L_a - 2a^2\rho_A + L_c\rho_B + L_b\rho_C, \tag{3}$$

and g_b, g_c are obtained cyclically.

Proof. Subtract the three power equations $|P-A|^2 - \rho_A = |P-B|^2 - \rho_B = |P-C|^2 - \rho_C$ and solve the resulting two linear equations. Conversion from Cartesian to barycentric coordinates gives (3). When all ρ vanish, $(g_a : g_b : g_c)$ reduces to the circumcenter $(a^2L_a : b^2L_b : c^2L_c)$, as it must. Moreover $g_a + g_b + g_c = 16\Delta^2 > 0$, independently of the power weights, where Δ is the triangle area. Thus this radical-center rule always represents a finite point on every nondegenerate face. □

3 A general positive edge-law construction

The following theorem organizes the positive-parameter constructions. The signed power families, treated afterwards, require additional domain arguments. The theorem supplies a common proof mechanism; no claim of priority over every possible edge-law construction is made.

Theorem 3.1 (Monotone homogeneous edge laws)**.** *Let $F : (0,\infty)^2 \to (0,\infty)$ be symmetric, real analytic, homogeneous of degree one, with both first partial derivatives strictly positive. Admit only nondegenerate Euclidean tetrahedra of the form*

$$e_{ij} = F(r_i, r_j), \qquad r_i > 0.$$

On the triangular domain where positive parameters are recovered from

$$a = F(r_B, r_C), \quad b = F(r_C, r_A), \quad c = F(r_A, r_B),$$

they are unique and real analytic. For every $t \neq 0$, define the Cevian rule $(r_A^t : r_B^t : r_C^t)$, and define the pedal rule by the radical weights (r_A^2, r_B^2, r_C^2). On tetrahedra whose faces lie in this domain, each rule selects exactly the displayed edge-law family. All Cevian centers and concurrence points are finite. The family is four-dimensional, including scale.

Proof. For the map taking (r_1, r_2, r_3) to the three edge values, its Jacobian in edge order $(12, 13, 23)$ is

$$\begin{pmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{pmatrix}, \qquad a, b, c, d, e, f > 0.$$

Its determinant is $-ade - bcf < 0$. Thus the inverse-function theorem gives a local analytic inverse. For global uniqueness, if a second triple had a larger first parameter, equality of edges $12, 13$ would force both other parameters to be smaller, contradicting equality of edge 23. Relabeling treats any unequal coordinate. Symmetry and homogeneity of the inverse give valid triangle-center rules.

For Cevian concurrence, the forward implication uses global weights r_i^t. Conversely, positivity permits positive global weights W_i. Recovered parameters in a face have the form $r_i^{(H)} = d_H W_i^{1/t}$, with $d_H > 0$. Homogeneity on a shared edge gives

$$e_{ij} = d_H F(W_i^{1/t}, W_j^{1/t});$$

its positive second factor makes the two adjacent face scalars equal. Connectivity gives a single global parameter vector.

For pedal concurrence the forward implication is the spatial power-center construction. Conversely, let the normals concur at P, and put $s_i = |P - V_i|^2$. Each face has a constant c_H such that $(r_i^{(H)})^2 = s_i - c_H > 0$. On a shared edge, the function

$$c \longmapsto F\big(\sqrt{s_i - c}, \sqrt{s_j - c}\big), \qquad c < \min(s_i, s_j),$$

is strictly decreasing. Hence the two constants agree, and the recovered radii are global. Finally, one face recovers three parameters and one additional incident edge recovers the fourth with nonzero derivative. The edge map therefore has a local inverse onto its image and rank four. Equal parameters give a regular tetrahedron; realizability remains open near it. This proves the dimension statement. □

For example,

$$F_\eta(r, s) = \sqrt{r^2 + s^2 + \eta r s}, \qquad \eta \geq 0,$$

satisfies the theorem. The values $\eta = 0, 1, 2$ recover the positive orthocentric, isogonic, and circumscriptible laws respectively. More generally the positive power law uses $F(r, s) = (r^k + s^k)^{1/k}$, and the Eisenstein law uses $F(r, s) = (r^{2k} + r^k s^k + s^{2k})^{1/(2k)}$. Thus coexistence of Cevian and pedal constructions alone is broad, not a rigidity condition. A sharper classification question should impose an additional feature, such as equality of the two planar centers, a prescribed algebraic degree, or a specified global domain.

4 The power-k family

Definition 4.1 (Power-k tetrahedron)**.** For $k \in \mathbb{R} \setminus \{0\}$, a tetrahedron belongs to $\mathcal{P}_k$ if real parameters p_1, p_2, p_3, p_4 exist such that

$$e_{ij}^k = p_i + p_j, \qquad e_{ij} = |V_i V_j|. \tag{4}$$

The six pair sums are therefore positive, and only parameter choices for which the resulting six lengths satisfy the simplex inequalities are admitted. The *positive branch* $\mathcal{P}_k^+$ consists of the members for which all $p_i > 0$.

Proposition 4.2 (Powered opposite-sum characterization). *A nondegenerate tetrahedron belongs to $\mathcal{P}_k$ if and only if*

$$e_{12}^k + e_{34}^k = e_{13}^k + e_{24}^k = e_{14}^k + e_{23}^k. \tag{5}$$

When these equations hold, the additive parameters p_i are unique.

Proof. Equation (4) makes every sum in (5) equal to $p_1 + p_2 + p_3 + p_4$. Conversely, set

$$p_1 = \frac{e_{12}^k + e_{13}^k - e_{23}^k}{2}, \quad p_2 = \frac{e_{12}^k + e_{23}^k - e_{13}^k}{2}, \quad p_3 = \frac{e_{13}^k + e_{23}^k - e_{12}^k}{2},$$

and $p_4 = e_{14}^k - p_1$. The two equalities in (5) give $e_{24}^k = p_2 + p_4$ and $e_{34}^k = p_3 + p_4$. Hence (4) holds. The first three edge equations already determine p_1, p_2, p_3, and then any edge incident with V_4 determines p_4, proving uniqueness. □

Since every e_{ij}^k is positive, at most one of the four p_i can be nonpositive. Under a similarity of ratio $r > 0$ the parameters scale as $p_i \mapsto r^k p_i$. Proposition 4.2 imposes two independent conditions on the six edge lengths, so the generic family has dimension four, including scale.

For a face ABC define

$$q_{k,a} = \frac{b^k + c^k - a^k}{2}, \qquad q_{k,b}, q_{k,c} \text{ cyclically.} \tag{6}$$

On a $\mathcal{P}_k$ face, with p_A, p_B, p_C attached to the vertices, one has

$$q_{k,a} = p_A, \qquad q_{k,b} = p_B, \qquad q_{k,c} = p_C. \tag{7}$$

Thus the facial defects recover the signed vertex parameters themselves. This three-term cancellation is the algebraic engine of the theory.

4.1 Cevian centers on signed chambers

For $x \neq 0$ and $t \in \mathbb{R}$ define the signed power

$$x^{\langle t \rangle} := \operatorname{sgn}(x)|x|^t. \tag{8}$$

It satisfies

$$(xy)^{\langle t \rangle} = x^{\langle t \rangle} y^{\langle t \rangle}, \qquad \left(x^{\langle t \rangle}\right)^{\langle 1/t \rangle} = x \quad (t \neq 0).$$

It is real analytic on each open half-line. Hence it is real analytic on the possibly disconnected open triangle domain

$$\Omega_k := \{(a, b, c) : q_{k,a} q_{k,b} q_{k,c} \neq 0\}, \tag{9}$$

where (a, b, c) ranges over nondegenerate Euclidean triangles. Let $\mathcal{U}_k$ be the open tetrahedral locus whose four faces lie in Ω_k. On a positive component the signed power is the ordinary real power. Define

$$C_{k,t} = (q_{k,a}^{\langle t \rangle} : q_{k,b}^{\langle t \rangle} : q_{k,c}^{\langle t \rangle}). \tag{10}$$

Theorem 4.3 (Signed power-k Cevian theorem). *Let $k \neq 0$ and $t \neq 0$. The rule $C_{k,t}$ is real analytic on Ω_k and*

$$\mathrm{ZCevian}[C_{k,t}] \cap \mathcal{U}_k = \mathcal{P}_k \cap \mathcal{U}_k.$$

Thus every generic signed branch of $\mathcal{P}_k$ is included, without requiring the four faces to have the same sign pattern. On the positive branch $\mathcal{P}_k^+$,

$$C_{k,t} \longrightarrow X(2) \qquad (t \to 0).$$

On any mixed-sign component of Ω_k, the same limit is the sign vector $(\operatorname{sgn} q_{k,a} : \operatorname{sgn} q_{k,b} : \operatorname{sgn} q_{k,c})$ *rather than the centroid.*

Proof. If $T \in \mathcal{P}_k$, then (7) makes the face coordinates in (10) proportional to $p_A^{\langle t\rangle}, p_B^{\langle t\rangle}, p_C^{\langle t\rangle}$. Lemma 2.2 applies with global weights $W_i = p_i^{\langle t\rangle}$.

Conversely, suppose the $C_{k,t}$-cevians concur. By the global-weight criterion, on every face F there is a nonzero face scalar λ_F such that

$$(q_{k,i}^{(F)})^{\langle t\rangle} = \lambda_F W_i \qquad (V_i \in F).$$

Applying the inverse signed power gives

$$q_{k,i}^{(F)} = c_F u_i, \qquad c_F = \lambda_F^{\langle 1/t\rangle}, \quad u_i = W_i^{\langle 1/t\rangle}.$$

For an edge $V_iV_j \subset F$, identity (6) gives

$$e_{ij}^k = q_{k,i}^{(F)} + q_{k,j}^{(F)} = c_F(u_i + u_j).$$

Writing the same edge from the adjacent face F' gives $c_F(u_i + u_j) = c_{F'}(u_i + u_j)$. The common value is $e_{ij}^k > 0$, so $u_i + u_j \neq 0$ and therefore $c_F = c_{F'}$. Connectivity of the face-adjacency graph makes all four face scalars equal. Absorbing their common value into the u_i yields (4). Finally, on the positive branch $q^{\langle t\rangle} = q^t \to 1$, giving $X(2)$. □

Remark 4.4 (Finite concurrence is a stricter assertion)**.** Theorem 4.3 is projective. Its finite version is obtained by imposing

$$\sum_i p_i^{\langle t\rangle} \neq 0, \qquad \sum_{j\neq i} p_j^{\langle t\rangle} \neq 0 \quad \text{for every } i.$$

These conditions hold automatically on the positive branch. They cannot be omitted on signed branches. For example, take

$$k = 2, \quad t = -\tfrac{1}{2}, \quad (p_1, p_2, p_3, p_4) = (-\tfrac{1}{9}, 1, 1, 1).$$

The three edges from V_1 have squared length $8/9$, the other three have squared length 2, and $V^2 = 1/54 > 0$. The global weights are $(-3, 1, 1, 1)$. All facial centers are finite, but the four cevians are parallel because the global sum is zero. Thus a finite-affine reading of the unqualified signed equality would be false. The positive-branch classical identifications are unaffected.

Thus each generic signed $\mathcal{P}_k$ has infinitely many chamberwise real-analytic Cevian representatives. A convenient reciprocal member is

$$X_k^{\mathrm{C}} := C_{k,-1} = \left(\frac{1}{b^k + c^k - a^k} : \frac{1}{c^k + a^k - b^k} : \frac{1}{a^k + b^k - c^k}\right), \tag{11}$$

where the harmless common factor 2 has been suppressed. When a single defect vanishes, the projectively equivalent product form

$$(q_{k,b}q_{k,c} : q_{k,c}q_{k,a} : q_{k,a}q_{k,b})$$

provides the natural finite continuation; such zero-parameter loci are lower-dimensional and are not part of the generic chamber statement above.

4.2 Pedal centers and the circumcenter endpoint

The pedal construction must have Euclidean weight degree two. Accordingly put

$$\rho_{k,a} = q_{k,a}^{\langle 2/k\rangle}, \qquad \rho_{k,b}, \rho_{k,c} \text{ cyclically}, \tag{12}$$

and let $Y_k = (g_{k,a} : g_{k,b} : g_{k,c})$, where g is given by (3) with $\rho = \rho_k$. On $\mathcal{P}_k^+$ this is simply $\rho_{k,a} = q_{k,a}^{2/k}$; if $r_i = p_i^{1/k} > 0$, then $\rho_i = r_i^2$ is an ordinary squared radius.

Theorem 4.5 (Power-k pedal theorem)**.** *Let $k \neq 0$. The rule Y_k is real analytic on Ω_k and satisfies the following statements.*

(i) *Every member of $\mathcal{P}_k \cap \mathcal{U}_k$ has the Y_k-pedal concurrence property.*

(ii) *If $k > 0$, the converse holds throughout the nonzero signed domain:*

$$\mathrm{ZPedal}[Y_k] \cap \mathcal{U}_k = \mathcal{P}_k \cap \mathcal{U}_k.$$

(iii) *For arbitrary $k \neq 0$, the converse holds on the positive component; thus the positive family selected by Y_k is exactly $\mathcal{P}_k^+$.*

If $O = (o_a : o_b : o_c) = X(3)$, where $o_a = a^2 L_a$, and $S_o = o_a + o_b + o_c$, $S_g = g_{k,a} + g_{k,b} + g_{k,c}$, then every fixed $\lambda \neq 0$ gives another representative

$$D_{k,\lambda} = ((1-\lambda) o_a S_g + \lambda g_{k,a} S_o : \textit{cyclic}). \tag{13}$$

The same inclusion/equality statements hold for $D_{k,\lambda}$, and $D_{k,\lambda} \to X(3)$ as $\lambda \to 0$.

Proof. First let $T \in \mathcal{P}_k$. Assign to V_i the real power weight

$$\rho_i = p_i^{\langle 2/k \rangle}.$$

There is a unique spatial power center P for which $|P - V_i|^2 - \rho_i$ is independent of i. On each face the three weights are exactly (12) by (7), and orthogonal projection of P to that face is their radical center. Lemma 2.3 therefore identifies every foot with Y_k. On the positive branch these weights are the squared radii of genuine vertex-centered spheres.

Conversely, suppose the normals at the four Y_k concur at P. Put $s_i = |P - V_i|^2$. On each face F there is a constant c_F such that

$$\rho_i^{(F)} = s_i - c_F \qquad (V_i \in F).$$

By (12),

$$q_{k,i}^{(F)} = (s_i - c_F)^{\langle k/2 \rangle}.$$

Hence for a shared edge $V_i V_j$,

$$e_{ij}^k = (s_i - c_F)^{\langle k/2 \rangle} + (s_j - c_F)^{\langle k/2 \rangle}. \tag{14}$$

If $k > 0$, the signed power $x \mapsto x^{\langle k/2 \rangle}$ is continuous and strictly increasing on $\mathbb{R}$, so the right-hand side of (14) is strictly decreasing in c_F. The two adjacent faces therefore have the same constant. If $k < 0$ but all defects are positive, then $s_i - c_F > 0$ and differentiation gives

$$-\frac{k}{2}\left((s_i - c_F)^{k/2-1} + (s_j - c_F)^{k/2-1}\right) > 0.$$

Thus the edge expression is again strictly monotone in c_F. Connectivity gives one constant c, and

$$p_i = (s_i - c)^{\langle k/2 \rangle}$$

satisfies (4).

For the pencil, normalize the barycentric triples of O and Y_k before taking the Euclidean affine combination $(1-\lambda)O + \lambda Y_k$; clearing denominators gives (13). Orthogonal projection is affine, so the spatial point $(1-\lambda)O_T + \lambda P$ projects to $D_{k,\lambda}$ on every face. Conversely, for $\lambda \neq 0$ one recovers P by subtracting the circumcenter component. □

Remark 4.6 (Negative exponents and mixed signs)**.** For $k < 0$ the forward signed construction remains valid, but a global converse on mixed-sign chambers cannot be obtained from monotonicity alone. The inverse signed power has singular walls. For example, at $k = -2$ the edge expression is $1/(s_i - c) + 1/(s_j - c)$, which can take the same positive value for constants in different sign intervals. The positive chamber has no such ambiguity and is the canonical domain for the negative-exponent statements in this paper.

Theorems 4.3 and 4.5 exhibit the promised pair of explicit center constructions depending on k. The Cevian theory is genuinely signed for every exponent. The pedal theory is genuinely signed for $k > 0$ and remains exact on $\mathcal{P}_k^+$ for all $k \neq 0$. The universal centers $X(2)$ and $X(3)$ occur as parameter endpoints on the positive branch but do not themselves select a proper four-dimensional family.

5 Classical members and the isodynamic limit

5.1 $k = 1$: circumscriptible tetrahedra or balloons

At $k = 1$, (4) becomes

$$e_{ij} = p_i + p_j.$$

Although the definition allows signed parameters, Euclidean realizability forces positivity in this case. Indeed, in any face containing the vertex V_i the triangle inequality

$$(p_i + p_j) + (p_i + p_\ell) > p_j + p_\ell$$

gives $p_i > 0$. Thus $\mathcal{P}_1 = \mathcal{P}_1^+$.

The law $e_{ij} = p_i + p_j$ is the classical balloon condition: there is a sphere tangent to all six edges, and p_i is the common tangent length from V_i to the three points of tangency on its incident edges [9, 10]. For a face ABC, let $s = (a + b + c)/2$ be its semiperimeter. Since $q_{1,a} = s - a$, the canonical Cevian center is

$$X_1^{\mathrm{C}} = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c} \right) = X(7),$$

the Gergonne point. Substitution of $\rho_a = (s - a)^2$ into (3) gives $Y_1 = (a : b : c) = X(1)$, the incenter. Hence

$$\mathrm{ZCevian}[X(7)] = \mathrm{ZPedal}[X(1)] = \mathcal{P}_1.$$

The pedal pencil (13) is the fixed-affine OI pencil from $X(3)$ to $X(1)$.

5.2 $k = 2$: the full orthocentric family

For $k = 2$, the edge law is

$$e_{ij}^2 = p_i + p_j$$

with the p_i allowed to have either sign. It immediately implies

$$e_{12}^2 + e_{34}^2 = e_{13}^2 + e_{24}^2 = e_{14}^2 + e_{23}^2,$$

the standard metric characterization of an orthocentric tetrahedron. Conversely, the equal-opposite-sum condition reconstructs real additive parameters p_i with $e_{ij}^2 = p_i + p_j$. Therefore

$$\boxed{\mathcal{P}_2 = \{\text{all orthocentric tetrahedra}\}},$$

including the branches for which one additive parameter is negative.

Furthermore

$$X_2^{\mathrm{C}} = \left(\frac{1}{b^2 + c^2 - a^2} : \text{cyclic} \right) = X(4),$$

and because $2/k = 1$, the signed pedal weight is simply $\rho_{2,a} = q_{2,a}$. Direct simplification of (3) again gives $Y_2 = X(4)$. Thus, with the usual limiting interpretation on right-face strata,

$$\mathrm{ZCevian}[X(4)] = \mathrm{ZPedal}[X(4)] = \mathcal{P}_2.$$

The pedal pencil is the Euler line $X(3)X(4)$; for example, its fixed-affine member $X(2)$ also selects the orthocentric family by pedal concurrence.

5.3 $k \to 0$: the normalized positive-branch isodynamic limit

There is no canonical $k \to 0$ limit if the signed additive parameters p_i are held fixed. The natural continuation is instead taken on the positive branch by writing

$$p_i = \frac{u_i^k}{2}, \qquad u_i > 0.$$

Then (4) becomes the power-mean law

$$e_{ij} = M_k(u_i, u_j) := \left(\frac{u_i^k + u_j^k}{2} \right)^{1/k}. \tag{15}$$

Then

$$\lim_{k \to 0} M_k(u_i, u_j) = \sqrt{u_i u_j}.$$

Consequently the limiting edges satisfy

$$e_{12}e_{34} = e_{13}e_{24} = e_{14}e_{23},$$

which is the classical isodynamic family $\mathcal{P}_0$.

The Cevian limit is singular in the two parameters. Put $\widehat{q}_{k,a} = b^k + c^k - a^k$. Since

$$\log \widehat{q}_{k,a} = k \log \frac{bc}{a} + O(k^2),$$

the choice $t = -1/k$ in (10) gives

$$\widehat{q}_{k,a}^{\,-1/k} : \widehat{q}_{k,b}^{\,-1/k} : \widehat{q}_{k,c}^{\,-1/k} \longrightarrow \frac{a}{bc} : \frac{b}{ca} : \frac{c}{ab} = a^2 : b^2 : c^2.$$

Thus the isodynamic limit has Cevian center $X(6)$, the symmedian point, in agreement with the classical theorem.

For the pedal limit, which is likewise a positive-branch limit, one must choose a regular representative of the affine pencil. Replacing $q_{k,a}$ in (12) by $\widehat{q}_{k,a}$ merely multiplies all squared radii by a common k-dependent factor, hence moves the center along the same $X(3)$-pencil without changing its nontrivial Z-pedal family. Now

$$\widehat{\rho}_{0,a} := \lim_{k \to 0} \widehat{q}_{k,a}^{\,2/k} = \left(\frac{bc}{a} \right)^2.$$

Substitution into (3) gives precisely the ETC center $X(58212)$ [13]. In the ETC it is described as the radical center of the circles centered at A, B, C with radii $bc/a, ca/b, ab/c$.

The limiting formulas give the correct candidate centers, but equality of zero sets is better proved directly. First suppose that

$$e_{12}e_{34} = e_{13}e_{24} = e_{14}e_{23}.$$

Then there are positive vertex parameters u_i with $e_{ij}^2 = u_i u_j$. On a face ABC the symmedian barycentrics are $(a^2 : b^2 : c^2) = (u_B u_C : u_C u_A : u_A u_B)$, which are proportional to $(1/u_A : 1/u_B : 1/u_C)$. Lemma 2.2 gives concurrence. Conversely, if the four $X(6)$-cevians concur, write the face scale opposite V_ℓ as λ_ℓ. The global-weight criterion gives, on the face ijk,

$$e_{jk}^2 = \lambda_\ell W_i, \qquad e_{ki}^2 = \lambda_\ell W_j, \qquad e_{ij}^2 = \lambda_\ell W_k.$$

Comparing a shared edge in its two adjacent faces shows that λ_m / W_m is independent of m. Hence all three products of opposite squared edges equal the same multiple of $W_1 W_2 W_3 W_4$, proving the isodynamic condition.

For the pedal direction, if $e_{ij}^2 = u_i u_j$ with $u_i > 0$, then on a face ABC

$$\frac{bc}{a} = u_A, \qquad \frac{ca}{b} = u_B, \qquad \frac{ab}{c} = u_C.$$

Thus the three squared power weights defining $X(58212)$ are the restrictions of the four global weights u_i^2. The spatial power center of these four weights projects orthogonally to $X(58212)$ on every face, proving pedal concurrence.

For the pedal converse, let the four normals through $X(58212)$ concur at P and put $s_i = |P - V_i|^2$. On a face F write

$$r_A = \frac{bc}{a}, \qquad r_B = \frac{ca}{b}, \qquad r_C = \frac{ab}{c}.$$

Since $X(58212)$ is the radical center for the squared weights r_i^2, there is a face constant c_F such that $r_i^2 = s_i - c_F$. For every edge $V_iV_j \subset F$, the three definitions above give $r_i r_j = e_{ij}^2$, and therefore

$$e_{ij}^4 = r_i^2 r_j^2 = (s_i - c_F)(s_j - c_F).$$

Here $c_F < \min(s_i, s_j)$, and the right-hand side is strictly decreasing in c_F. The two face constants adjacent to every edge are therefore equal; connectivity gives one constant c. Setting $r_i = \sqrt{s_i - c} > 0$ yields $e_{ij}^2 = r_i r_j$, and hence the three products of opposite edges are equal. Thus the limiting identities are exact:

$$\boxed{\mathrm{ZCevian}[X(6)] = \mathrm{ZPedal}[X(58212)] = \mathcal{P}_0.} \tag{16}$$

6 A handful of exact ETC identifications

The formulas above meet several published barycentric functions in Kimberling's Encyclopedia of Triangle Centers (ETC) [13]. A database census is not needed here. Instead we record five representative identities and display the actual first-coordinate factorization in each case. Put

$$\widetilde{q}_{k,a} = b^k + c^k - a^k$$

and obtain the other two coordinates cyclically. Common nonzero factors are projectively irrelevant.

Table 1: Five exact ETC identifications. The last column displays the first barycentric coordinate as it appears algebraically and its reduction to the power-k template.

k	ETC center	first-coordinate identity
0	$X(6)$	$f_a = a^2$, the normalized $k \to 0$ Cevian limit
1	$X(7)$	$f_a = \dfrac{1}{s-a} = \dfrac{2}{b+c-a} = 2\widetilde{q}_{1,a}^{-1}$
2	$X(4)$	$f_a = \dfrac{1}{b^2+c^2-a^2} = \widetilde{q}_{2,a}^{-1}$
-3	$X(17486)$	$f_a = a^3b^3 + a^3c^3 - b^3c^3 = (abc)^3\widetilde{q}_{-3,a}$
$\frac{3}{2}$	$X(20346)$	$f_a = a^{3/2} - b^{3/2} - c^{3/2} = -\widetilde{q}_{3/2,a}$

The first three rows are the classical isodynamic, balloon, and orthocentric members. The last two show why catalogue recognition should be performed projectively rather than by looking only for the reciprocal representative $C_{k,-1}$: an ETC entry may differ by a cyclic common factor or use a different nonzero power of the same separable defect. The pedal side has the three exact classical identifications already proved above,

$$\mathrm{ZPedal}[X(58212)] = \mathcal{P}_0, \qquad \mathrm{ZPedal}[X(1)] = \mathcal{P}_1, \qquad \mathrm{ZPedal}[X(4)] = \mathcal{P}_2.$$

Their fixed-affine pencils with $X(3)$ are, respectively, the natural $X(3)X(58212)$ line, the OI line, and the Euler line. Broader ETC enumeration belongs to a separate computational study and is not used in this article.

7 Three-dimensional concurrence plots

For the following plots the base is the 13–20–21 triangle

$$A = (0, 0, 0), \qquad B = (21, 0, 0), \qquad C = (16, 12, 0),$$

whose area is 126. The apex was reconstructed by trilateration from the six power-law edges. Dashed lines are the cevians or face normals, orange points are the four face centers, and the red point is the spatial concurrence point.

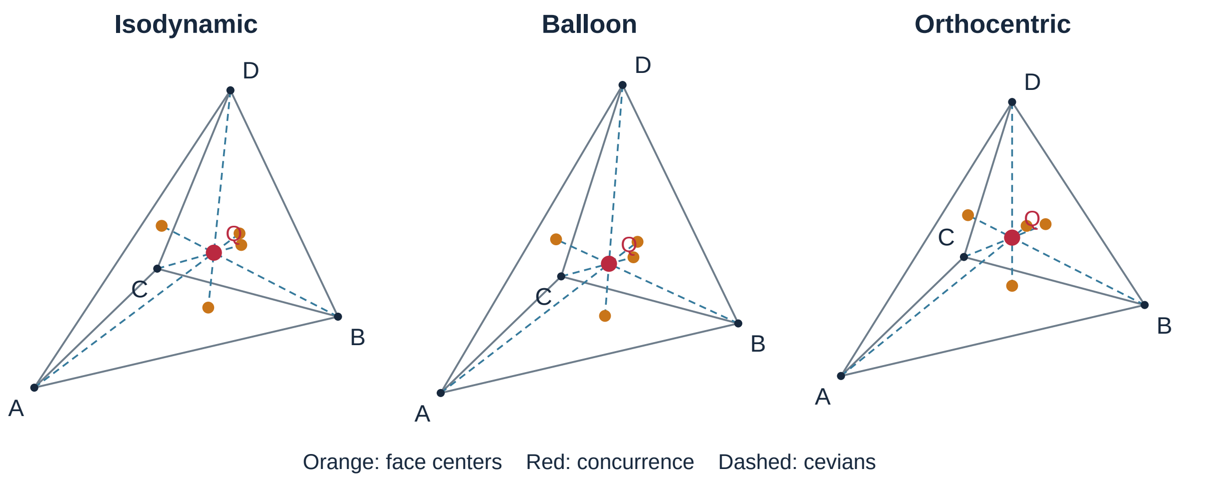

Figure 1: Cevian concurrence for the isodynamic limit, balloons, and orthocentric tetrahedra. The displayed examples use $u_D = 16$ at $k = 0$, $p_D = 9$ at $k = 1$, and $p_D = 100$ at $k = 2$.

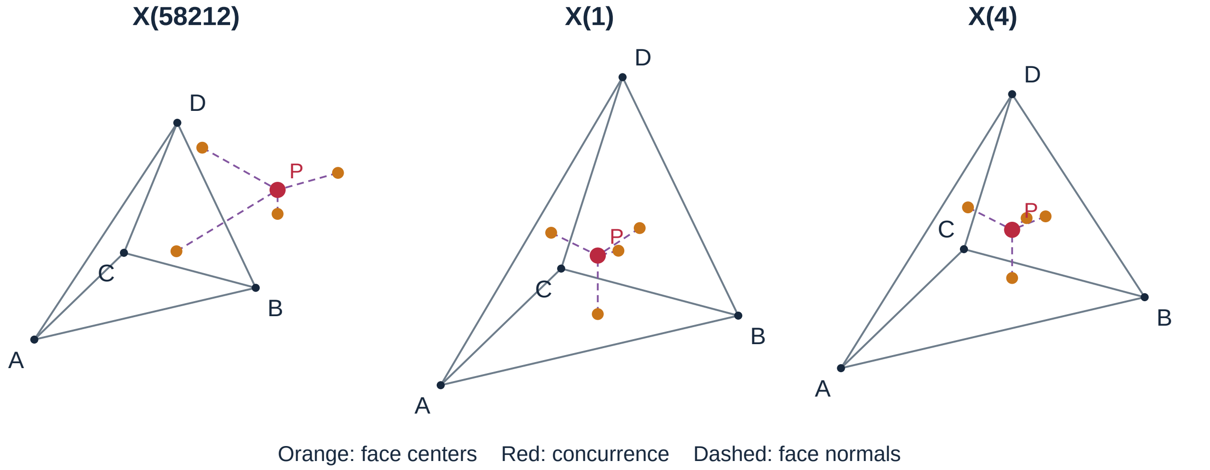

Figure 2: Pedal concurrence for the same three families. At $k = 0$ the displayed representative is the regularized center $X(58212)$; at $k = 1, 2$ they are $X(1)$ and $X(4)$.

For example, the exact base parameters in the revised additive convention are

case	vertex parameters at A, B, C
$k=0$	$(u_A, u_B, u_C) = \left(\frac{420}{13}, \frac{273}{20}, \frac{260}{21}\right)$
$k=1$	$(p_A, p_B, p_C) = (14, 7, 6)$
$k=2$	$(p_A, p_B, p_C) = (336, 105, 64)$.

They reproduce the same 13–20–21 base geometry and provide convenient checks for independent computer algebra and dynamic geometry.

8 Residual tests and dimension certificates

For real-analytic center functions, the formulas below are exact compatibility tests and local rank tools. They are not a terminating global algorithm. For polynomial or rational data, elimination and saturation give a separate algebraic certification procedure; real realizability must still be checked.

The six edge lengths form local coordinates on the open cone of nondegenerate tetrahedra. Concurrence can be tested without embedding the tetrahedron in $\mathbb{R}^3$. This substantially accelerates large ETC searches.

For a face opposite V_ℓ, let $w_i^{(\ell)}$ be the barycentric weight assigned by f to vertex V_i of that face. If $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$, define

$$C_{ij} = w_i^{(k)} w_j^{(\ell)} - w_j^{(k)} w_i^{(\ell)}. \tag{17}$$

Proposition 8.1 (Cevian residual test)**.** *Away from zero barycentric coordinates, the four Z-cevians concur if and only if the six residuals C_{ij} vanish.*

Proof. Equation (17) says that the ratio of the weights at the two ends of every edge is the same when computed in either adjacent face. These compatible ratios assemble into four global weights. Lemma 2.2 then proves concurrence. The converse is immediate from the same lemma. □

There is an equally short metric test for pedal concurrence. Normalize the face weights to $\lambda_i^{(\ell)} = w_i^{(\ell)} / \sum_{m\neq\ell} w_m^{(\ell)}$, put $d_{ij} = e_{ij}^2$ and $d_{ii} = 0$, and define

$$\pi_{ij}^{(\ell)} = \sum_{m\neq\ell} \lambda_m^{(\ell)} (d_{mj} - d_{mi}), \qquad P_{ij} = \pi_{ij}^{(k)} - \pi_{ij}^{(\ell)}. \tag{18}$$

Proposition 8.2 (Pedal residual test)**.** *The four normals at the face centers concur if and only if the six residuals P_{ij} vanish.*

Proof. If $Z_\ell = \sum_{m\neq\ell} \lambda_m^{(\ell)} V_m$, then, up to the face-independent term $|V_i|^2 - |V_j|^2$, the quantity $2Z_\ell \cdot (V_i - V_j)$ is exactly $\pi_{ij}^{(\ell)}$. Hence $P_{ij} = 0$ says that the two adjacent faces prescribe the same tangential equation along their common edge. On every triangular face the three edge equations have zero cyclic sum; consequently the six compatible edge values integrate to one affine linear functional, represented by a unique spatial point. Its projections are the four prescribed face centers. □

Let R denote either residual vector and let $J = \partial R / \partial(e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$. At a regular solution on a fixed component,

$$\dim_{\text{local}} = 6 - \operatorname{rank} J. \tag{19}$$

The efficient workflow is therefore: form the six residuals; clear denominators and introduce minimal-polynomial variables for algebraic radicals; find one admissible point on every component;

evaluate the Jacobian at high precision; and certify promising components by exact rank or by a saturated Gröbner basis. One must not estimate dimension at a singular or equilateral point, where the rank can drop artificially.

Formula (19) is a local statement at a regular point. A global dimension claim additionally requires all irreducible components to be controlled; for algebraic rules this can be certified after denominator saturation by a Gröbner basis or Hilbert polynomial. Real simplex inequalities and components created by cleared denominators must then be checked separately. In particular, a rank calculation near the regular tetrahedron cannot exclude a remote component of larger dimension.

8.1 Dimension: scope and current status

The residual test separates a local regularity statement from a global classification. For $\mathcal{P}_k$, Proposition 4.2 gives two independent relations among the six positive edge coordinates, while $e \mapsto e^k$ is a local analytic diffeomorphism on $e > 0$. Hence $\mathcal{P}_k$ has generic dimension four, including scale. The Eisenstein family $\mathcal{E}_k$ is also generically four-dimensional, but for a different reason: on the Fermat-admissible domain introduced below, the auxiliary triangle reconstructs u_A, u_B, u_C uniquely and analytically. One additional incident edge reconstructs the positive fourth parameter explicitly; for example,

$$u_D = \frac{-u_A + \sqrt{4e_{AD}^{2k} - 3u_A^2}}{2}.$$

Thus its four-parameter edge map has a local analytic inverse. Neither conclusion is a Jacobian-rank guess.

For arbitrary center rules the possible dimensions are subtler. The universal two-dimensional kite locus excludes global dimension one [8]. The companion face-constraint paper proves that a proper globally real-analytic Z-Cevian family with nowhere-zero facial barycentric weights has no five-dimensional analytic component, and it constructs globally real-analytic rules having genuine three-dimensional components (companion D). This does *not* yet produce a center whose entire global Z-Cevian family is proved to have dimension exactly three: a separate remote four-dimensional component must also be excluded. Thus the rigorous status is

$$\begin{aligned} &\text{dimension } 1: && \text{impossible globally,}\\ &\text{dimension } 5: && \text{excluded under the nonvanishing-weight hypothesis,}\\ &\text{dimension } 3: && \text{analytic components occur.}\end{aligned}$$

What remains uncertified is an explicit center whose *entire* global Z-Cevian family has dimension exactly three. This distinction is also why the chamberwise analytic rules in the present paper must not be silently substituted into a theorem whose hypothesis is global real analyticity.

9 The Eisenstein-power superfamily

Definition 9.1. For $k \neq 0$, let $\mathcal{E}_k$ be the family

$$e_{ij}^{2k} = u_i^2 + u_i u_j + u_j^2, \tag{20}$$

with positive parameters and admissible simplex inequalities.

The name comes from the norm form $x^2 + xy + y^2$. The elementary geometry behind it is worth making explicit.

Lemma 9.2 (The 120° realization). *Let $x, y, z > 0$. From a point F draw three rays whose pairwise angles are $120°$, and place A, B, C on them at distances x, y, z, respectively. Then*

$$a^2 = y^2 + yz + z^2, \quad b^2 = z^2 + zx + x^2, \quad c^2 = x^2 + xy + y^2.$$

Moreover, F is inside ABC and is its first Fermat point.

Proof. The three cosine-law identities are immediate. If v_A, v_B, v_C are the unit vectors along the rays, then $v_A + v_B + v_C = 0$. Since $A - F = xv_A$, $B - F = yv_B$, and $C - F = zv_C$, the positive barycentric weights $(1/x : 1/y : 1/z)$ have weighted vector sum zero. Thus F lies inside ABC. The three angles at F are $120°$, which is the Fermat–Torricelli characterization in the interior regime [3, 15]. □

On a face with sides a, b, c, put

$$A = a^k, \qquad B = b^k, \qquad C = c^k.$$

The three transformed lengths need not form a triangle for an arbitrary abc. Let $\mathcal{A}_k$ denote the open set on which they form a nondegenerate triangle whose angles are all less than $120°$, and let Δ_k be its area. The center rules below are defined on this natural Fermat-admissible domain. Every face arising from positive parameters in $\mathcal{E}_k$ lies in $\mathcal{A}_k$ by Lemma 9.2. Then

$$A^2 = u_B^2 + u_B u_C + u_C^2,$$

cyclically. By Lemma 9.2, u_A, u_B, u_C are the distances from the first Fermat point of the auxiliary triangle to its vertices. Conversely, the positive parameters are recovered uniquely from that triangle. Indeed, with $S = u_A + u_B + u_C$, area decomposition at the Fermat point gives

$$\Delta_k = \frac{\sqrt{3}}{4}(u_A u_B + u_B u_C + u_C u_A).$$

If

$$\Sigma_k = \frac{A^2 + B^2 + C^2 + 4\sqrt{3}\,\Delta_k}{2},$$

then substitution gives $\Sigma_k = S^2$ and hence

$$u_A = \frac{\Sigma_k + B^2 + C^2 - 2A^2}{3\sqrt{\Sigma_k}}, \tag{21}$$

cyclically. These formulas give a real-analytic inverse on $\mathcal{A}_k$ for the positive parameter triple attached to a face; in particular, the recovery is unique and stable under small admissible perturbations.

Define the explicit Fermat barycentric function

$$\Phi(A, B, C) = A^4 - 2(B^2 - C^2)^2 + A^2\big(B^2 + C^2 + 4\sqrt{3}\,\Delta_k\big). \tag{22}$$

This proportionality is exact: substituting the three norm expressions and the area formula above yields

$$\Phi(A, B, C) = 6u_B u_C(u_A + u_B + u_C)^2 = 6u_A u_B u_C(u_A + u_B + u_C)^2 \frac{1}{u_A}. \tag{23}$$

Thus the cyclic barycentric triple is proportional to $(1/u_A : 1/u_B : 1/u_C)$. The same function is, up to a common factor, $A\csc(\alpha + \pi/3)$, where α is opposite A in the auxiliary triangle. Set

$$E_k^{\mathrm{C}} = (\Phi(A, B, C) : \Phi(B, C, A) : \Phi(C, A, B)), \tag{24}$$

$$\sigma_{k,a} = u_A^{2/k}, \tag{25}$$

and define E_k^{P} by substituting $\rho = \sigma_k$ into (3) for the original triangle abc.

Theorem 9.3 (Eisenstein Cevian–pedal duality)**.** *Restrict to tetrahedra whose four faces lie in $\mathcal{A}_k$. On every positive admissible branch,*

$$\mathrm{ZCevian}[E_k^{\mathrm{C}}] = \mathrm{ZPedal}[E_k^{\mathrm{P}}] = \mathcal{E}_k.$$

The powers $(E_k^{\mathrm{C}})^t$, $t \neq 0$, give the same Cevian family, and the fixed-affine pencil between E_k^{P} and $X(3)$ gives the same pedal family except at its universal circumcenter endpoint.

Proof. On an $\mathcal{E}_k$ face, (23) supplies weights proportional to $1/u_i$. Lemma 2.2 proves Cevian concurrence. Conversely, let W_i be the global weights supplied by Lemma 2.2. On a face F, proportionality of its center coordinates to W_i implies $u_i^{(F)} = d_F/W_i$ for a positive face scalar d_F. Hence a shared edge satisfies

$$e_{ij}^{2k} = d_F^2 \left(W_i^{-2} + (W_i W_j)^{-1} + W_j^{-2} \right).$$

Writing the same edge from its other adjacent face F' gives $d_F^2 = d_{F'}^2$. Positivity and face connectivity make every d_F equal, and (20) follows with one global parameter per vertex.

For the pedal statement, place a sphere of radius $u_i^{1/k}$ at V_i. Its squared radius is (25), so Lemma 2.3 gives E_k^{P} on every face. In the converse, two adjacent faces can differ by an additive squared-radius constant c. For a shared edge the right side is

$$(s_i + c)^k + \left((s_i + c)(s_j + c)\right)^{k/2} + (s_j + c)^k,$$

On the interval where $s_i + c, s_j + c > 0$, its derivative is

$$k(s_i + c)^{k-1} + k(s_j + c)^{k-1} + \frac{k}{2}\left((s_i + c)(s_j + c)\right)^{k/2} \left(\frac{1}{s_i + c} + \frac{1}{s_j + c} \right).$$

All terms have the sign of k, so the derivative never vanishes. The expression is strictly monotone for every $k \neq 0$ on the positive branch. Hence all face constants agree and (20) follows. The two one-parameter extensions are proved as in Theorems 4.3 and 4.5. □

9.1 Why $X(13)$ is special

Throughout this subsection, every face is assumed to have all angles strictly less than 120°, as required by $\mathcal{A}_1$. At $k = 1$, the auxiliary triangle is the original triangle, and (20) is the classical isogonic law

$$e_{ij}^2 = u_i^2 + u_i u_j + u_j^2.$$

Equations (24) and (25) both reduce to the first Fermat point $X(13)$. Therefore

$$\mathrm{ZCevian}[X(13)] = \mathrm{ZPedal}[X(13)] = \mathcal{E}_1. \tag{26}$$

The published ETC coordinate functions also give the exact projective identities

$$f_{298} \sim f_{13}^{-1}, \qquad f_{11080} \sim f_{13}^{2}, \qquad f_{11129} \sim f_{13}^{-2}.$$

The global-weight criterion is unchanged by a nonzero power, so these four entries are representatives of one Cevian family:

$$\mathrm{ZCevian}[X(13)] = \mathrm{ZCevian}[X(298)] = \mathrm{ZCevian}[X(11080)] = \mathrm{ZCevian}[X(11129)] = \mathcal{E}_1.$$

This is a direct functional identity, not a conclusion from a finite numerical sample. For comparison, $X(15222)$ and $X(15225)$ encode one further four-parameter min–max family, but their functions involve min and max and are not real analytic along an isosceles wall. They therefore do not compete with the analytic Eisenstein construction discussed here.

Terminology requires care here. The Fermat–Torricelli point of a tetrahedron is normally the geometric median [3]

$$M = \arg\min_{X\in\mathbb{R}^3} \sum_{i=1}^{4} |X - V_i|.$$

When M is not a vertex, it satisfies the vector equilibrium $\sum_i (V_i - M)/|V_i - M| = 0$. This minimization definition is not, in general, equivalent to requiring equal pairwise ray angles or equal solid angles with the faces. Such equal-angle constructions define separate isogonic centers and agree with the geometric median only in special symmetric cases.

The 2020 concurrence paper calls the intersection of the cevians to the four facial Fermat–Torricelli points the $X(13)$-cevian center, not the tetrahedral Fermat–Torricelli point [8]. We denote it by Q and call it the *Fermat-cevian concurrence point.* Let P denote instead the pedal concurrence point supplied by Theorem 9.3. For $k = 1$ its projection onto a face is the 120° point of Lemma 9.2, whose distances to the three face vertices are the corresponding u_i. Hence, for a face F,

$$|P - V_i|^2 - u_i^2 = d(P, F)^2 \qquad (V_i \in F).$$

The left-hand side is the same spatial power for all vertices, so $d(P, F)$ is the same for all four faces. To verify interiority rather than assume it, put

$$A_i = \sum_{\substack{j<\ell \\ j,\ell\neq i}} u_j u_\ell > 0.$$

The face-area formula gives $\Delta_i = \sqrt{3}A_i/4$. Writing $d_{ij} = u_i^2 + u_i u_j + u_j^2$, $d_{ii} = 0$, expansion gives

$$\sum_\ell A_\ell (d_{\ell j} - d_{\ell i}) = (u_j^2 - u_i^2) \sum_\ell A_\ell.$$

Thus the point with barycentrics $(A_1 : A_2 : A_3 : A_4)$ has the required power differences, and is the unique power center P. These are positive face-area weights, so P is exactly the tetrahedral incenter I. Equivalently, the insphere contact points are the facial Fermat points, a standard characterization of isogonic tetrahedra [8]. Thus $X(13)$ is self-dual with respect to the two face-center mechanisms, even though Q and I are generally different.

9.2 An exact 13–20–21 isogonic example

Take

$$A = (0,0,0), \quad B = (21,0,0), \quad C = (16,12,0), \qquad \Sigma = 505 + 252\sqrt{3}, \qquad \sigma = \sqrt{\Sigma},$$

and attach to the base vertices the Fermat distances

$$u_A = \frac{\Sigma + 503}{3\sigma}, \qquad u_B = \frac{\Sigma - 190}{3\sigma}, \qquad u_C = \frac{\Sigma - 313}{3\sigma}.$$

Choose $u_D = u_C$. Then the isogonic edge law yields $AD = 20$, $BD = 13$ and $CD = \sqrt{3}\,u_C$. A particularly simple exact apex is

$$D = \left(16,\ 12 - \frac{u_C^2}{8},\ \sqrt{144 - \left(12 - \frac{u_C^2}{8}\right)^2}\right) \doteq (16, 6.1731158272, 10.2904150054). \tag{27}$$

The two spatial points are specified exactly by their tetrahedral barycentrics:

$$Q = \left(\frac{1}{u_A} : \frac{1}{u_B} : \frac{1}{u_C} : \frac{1}{u_C}\right), \tag{28}$$

$$I = (F_A : F_B : 126 : 126), \tag{29}$$

$$F_A = \frac{\sqrt{3}}{4}(u_C^2 + 2u_B u_C), \qquad F_B = \frac{\sqrt{3}}{4}(u_C^2 + 2u_A u_C). \tag{30}$$

In Cartesian coordinates,

$$Q \doteq (15.1502684777, 5.5551292656, 3.1455577621),$$
$$I \doteq (14.7761700885, 5.2830577098, 2.9914989179),$$

and $|QI| \doteq 0.4875517036$. For comparison, the ordinary tetrahedral Fermat–Torricelli point obtained from the vector equilibrium is

$$M \doteq (15.1196608204, 5.5517194945, 3.1436270002), \qquad |QM| \doteq 0.0308574637.$$

It is close to Q in this symmetric example, but it is not Q and is not I.

No member of this isogonic family over the chosen base can have a fully rational apex. Indeed,

$$u_A - u_B = \frac{231}{\sigma}, \qquad u_A - u_C = \frac{272}{\sigma}, \qquad u_B - u_C = \frac{41}{\sigma}.$$

If the horizontal coordinates of D were rational, both $AD^2 - BD^2$ and $AD^2 - CD^2$ would be rational. The isogonic law would then make both $(u_A + u_B + u_D)/\sigma$ and $(u_A + u_C + u_D)/\sigma$ rational. Their difference is $41/\Sigma$, which is irrational, a contradiction. Thus no member of this isogonic family over the chosen rational base has a fully rational apex.

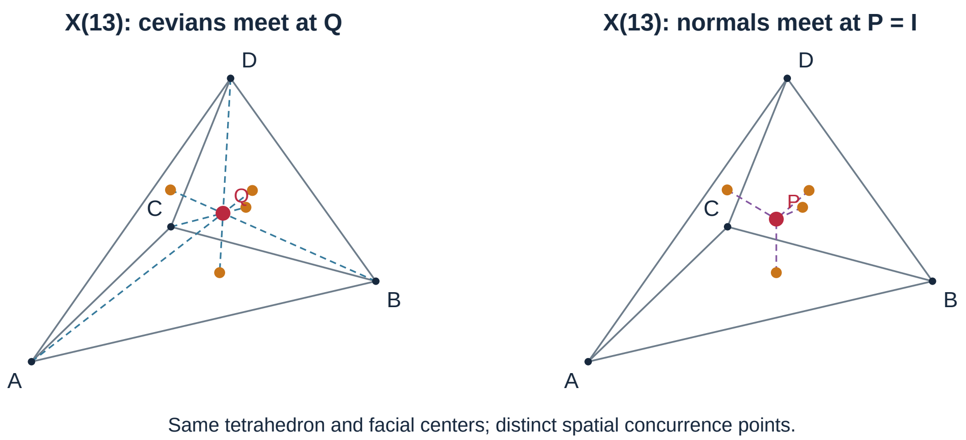

Figure 3: Vector diagram of the isogonic tetrahedron (27). The same four orange facial points $X(13)$ support the blue cevians through Q and the purple face normals through the incenter I.

This is the geometric connection that distinguishes the Eisenstein-power superfamily from an arbitrary deformation of $\mathcal{P}_k$: it possesses explicit and parallel Cevian and pedal centers for every exponent, and its isogonic member has a single planar center performing both jobs.

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10 Conclusions and open problems

The positive edge-law theorem separates the common construction from the special signed, limiting, and self-dual phenomena. The classical balloon, orthocentric, isodynamic, and Fermat concurrence characterizations have predecessors in [8]; the present contribution is the unified construction, its precise converse domains, and the residual framework.

The same distinguished members $\mathcal{P}_1$ and $\mathcal{P}_2$ reappear in the companion skeletal-centroid and Euler-projection classifications (companion S, E). Several questions remain open. The mixed-sign pedal converse for negative k requires a separate semialgebraic analysis because inverse-power singular walls can admit extra components. Theorem 3.1 shows that positive monotone homogeneous edge laws already support both constructions. A more selective problem is to classify self-dual laws, or laws subject to a prescribed algebraic degree and domain. Mere coexistence of the two constructions does not imply rigidity. A separate global problem is to construct a globally real-analytic center whose entire Z-Cevian family, rather than merely one component, is proved to have dimension three.

Minimal computational supplement

The exact-check suite is supplied as `independent_checks.py`. The original Wolfram source is also included, with its execution status documented in the README. The vector-diagram generator and its numerical incidence checks are supplementary illustrations, not proof certificates.

Computational materials

The accompanying bundle contains editable sources, build instructions, independent exact-check scripts and their recorded outputs. These support the displayed calculations; open global classification problems are not certified by local checks.

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