Skeletal-Centroid Projection Properties of Tetrahedra

Circumscriptible families, jaws, and a symmetry principle

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Abstract

A tetrahedron has three generally distinct centroids arising from uniform mass on its vertices, edges and surface; its volume centroid equals its vertex centroid. We classify tetrahedra for which the line through the vertex and edge centroids projects into the corresponding skeletal-centroid object on every triangular face. Both opposite-vertex central projection and orthogonal projection are considered, with collapsed lines treated as points. For central projection, the complete locus consists of the circumscriptible family together with jaws and equifacial tetrahedra having no equilateral face. The four-dimensional circumscriptible component is characterized by equality of the three opposite-edge sums. For orthogonal projection, the complete locus consists of jaws without equilateral faces and tetrahedral kites. The equilateral boundary is proved separately because the facial object collapses. A symmetry theorem explains the persistence of jaws for more general equivariant choices of centers. The same bilinear edge system appears in the Euler-projection problem with squared lengths in place of lengths, clarifying the relation between the two classifications. Explicit examples and exact factorizations accompany the proofs.

Keywords. Tetrahedron; skeletal centroid; Spieker center; cevian projection; orthogonal projection; circumscriptible tetrahedron.

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1 Introduction

Notation across the series. Throughout the series, $e_{ij} = |V_iV_j|$ denotes an edge length and $d_{ij} = e_{ij}^2$ its square. Dimensions include scale; passing to similarity classes subtracts one. Local six-variable aliases are specified at each metric calculation. Here $(x, y, z, u, v, w) = (e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$.

Equal point masses at the vertices of an n-simplex and uniform mass in its n-dimensional interior have the same centroid. Uniform mass on an intermediate skeleton generally gives a different point. Krantz, McCarthy, and Parks proved the equality of the zero- and top-dimensional centroids and studied the generic distinction of the intermediate skeletal centroids [9].

For a triangle, uniform boundary mass gives the Spieker center $X(10)$, whereas uniform area mass and equal vertex masses give $X(2)$. Thus $X(2)X(10)$ is the unique nontrivial line formed by its skeletal centroids. It is also the centroid–incenter line.

This article is one part of a four-paper study of how planar center geometry constrains tetrahedra. The power–k companion introduces

$$\mathcal{P}_k : \qquad e_{ij}^k = p_i + p_j$$

with signed vertex parameters and develops parallel Cevian–pedal concurrence theories (companion P). Its special members $\mathcal{P}_1$ and $\mathcal{P}_2$ are, respectively, the circumscriptible and full orthocentric families. A second companion studies lower-dimensional face constraints and analytic Z-cevian dimension obstructions (companion D).

The Euler-projection companion (companion E) asks a different question: when does the tetrahedral Euler object project into the Euler object of every face? Its maximal component is $\mathcal{P}_2$. Here the spatial object is instead the line from the ordinary centroid to the edge-skeleton centroid. The same four formal edge polynomials reappear, but now in ordinary rather than squared edge lengths, and the maximal cevian component becomes $\mathcal{P}_1$. This $\mathcal{P}_2/\mathcal{P}_1$ pairing is the central structural link between the two projection papers.

All dimensions below include scale. A labeled tetrahedron has dimension six, a circumscriptible tetrahedron dimension four, and a jaw dimension three. Passing to similarity classes subtracts one.

Companion manuscripts. The unpublished companion papers by the present author are designated (P) *Power-k tetrahedra: a unified Cevian–pedal theory of four-parameter tetrahedral families*; (D) *Dimensions of tetrahedral Z-cevian families: face constraints, jaws, tripods, and odd-dimensional obstructions*; (E) *Tetrahedra with orthogonal and cevian Euler–projection properties: edge-length classifications with orthocentric, jaw, equifacial, and diametral-mirror components.* All are version 2, dated 6 September 2026; these are local version identifiers, not public repository records. Results needed for the present proofs are stated and proved here.

2 Centroids supported in different dimensions

Let $T = ABCD$. For $k = 0, 1, 2, 3$, let $C_k(T)$ be the center of mass obtained from constant density on the union of all k-faces, using k-dimensional measure. For $k = 0$, this means equal point masses. Write

$$\begin{aligned} x &= AB, \quad y = AC, \quad z = AD, \\ u &= BC, \quad v = BD, \quad w = CD, \end{aligned} \tag{1}$$

and $L = x + y + z + u + v + w$.

Proposition 2.1 (Tetrahedral mass centers)**.**

$$C_0(T) = C_3(T) = G = \frac{A + B + C + D}{4}, \tag{2}$$

$$W = C_1(T) = \frac{(x + y + z)A + (x + u + v)B + (y + u + w)C + (z + v + w)D}{2L}. \tag{3}$$

Let $\Delta_A, \ldots, \Delta_D$ be the areas of the opposite faces and $\Sigma = \Delta_A + \Delta_B + \Delta_C + \Delta_D$. Then

$$Q = C_2(T) = \frac{\sum_{\mathrm{cyc}}(\Sigma - \Delta_A)A}{3\Sigma}. \tag{4}$$

If I is the tetrahedral incenter, then

$$\boxed{Q = \frac{4G - I}{3}}. \tag{5}$$

Proof. The first identity is the affine centroid formula for a simplex. Each edge contributes its length times its midpoint to the wire-frame first moment, which gives (3). Weighting each face centroid by its area gives (4). Finally,

$$I = \frac{\Delta_A A + \Delta_B B + \Delta_C C + \Delta_D D}{\Sigma};$$

combining this with $4G = A + B + C + D$ proves (5). □

The three generally distinct tetrahedral centers are

$$G = C_0 = C_3, \qquad W = C_1, \qquad Q = C_2.$$

Equation (5) also gives

$$Q - G = \frac{G - I}{3},$$

so, whenever the two points are distinct, the surface-centroid line is exactly the centroid–incenter line. We shall write

$$\ell_T^{GI} := \operatorname{aff}\{G, Q\} = \operatorname{aff}\{G, I\},$$

with the obvious point interpretation when the centers coincide.

For a face $F = ABC$, put

$$a = BC, \qquad b = CA, \qquad c = AB, \qquad p = a + b + c.$$

Proposition 2.2 (Planar mass centers)**.**

$$C_0(F) = C_2(F) = g = X(2) = \frac{A + B + C}{3},$$

and the boundary-wire centroid is

$$s = C_1(F) = X(10) = \frac{(b+c)A + (c+a)B + (a+b)C}{2p}. \tag{6}$$

If i is the face incenter, then

$$\boxed{s = \frac{3g - i}{2}}. \tag{7}$$

Hence g, s, i are collinear, more precisely

$$s - g = \frac{g - i}{2}.$$

We denote their common affine object (a point when the face is equilateral) by

$$\ell_F^{\mathrm{sk}} := \operatorname{aff}\{g, s\} = \operatorname{aff}\{g, i\}.$$

Proof. Weighting the side midpoints by the side lengths gives (6). Since $i = (aA + bB + cC)/p$, the coefficient of A in $(3g - i)/2$ is $(1 - a/p)/2 = (b + c)/(2p)$, and cyclically. $\square$

The identification $C_1(F) = X(10)$, with barycentrics $(b + c : c + a : a + b)$, is classical; see Kimberling [8, 2] and Honsberger [7].

Definition 2.3. The spatial skeletal-centroid object is

$$\mathcal{S}_T = \begin{cases} \operatorname{aff}\{G, W\}, & G \neq W, \\ \{G\}, & G = W. \end{cases}$$

The facial object is

$$\mathcal{S}_F = \begin{cases} \ell_F^{\mathrm{sk}}, & F \text{ non-equilateral}, \\ \{g\}, & F \text{ equilateral}. \end{cases}$$

3 Projection properties and the one-face lemma

For $F = ABC$ opposite D, let π_F be orthogonal projection. For a point $Y \neq D$, write

$$\rho_F(Y) = \operatorname{aff}\{D, Y\} \cap \operatorname{aff}(F)$$

whenever this intersection is a point; in particular this is always defined for $Y = G, W$. For the spatial skeletal-centroid object itself, central projection is defined directly as a line-object:

$$\rho_F(\mathcal{S}_T) := \operatorname{aff}(D, \mathcal{S}_T) \cap \operatorname{aff}(F). \tag{8}$$

Because $G \in \mathcal{S}_T$ and the median DG meets F at g, the intersection in (8) is always nonempty and is either a line or the single point g. Put

$$P = \pi_F(D), \qquad R = \rho_F(W).$$

Definition 3.1. T has the *cevian skeletal-centroid projection property* (CSCP) if $\rho_F(\mathcal{S}_T) \subseteq \mathcal{S}_F$ for all faces. It has the *orthogonal skeletal-centroid projection property* (OSCP) if $\pi_F(\mathcal{S}_T) \subseteq \mathcal{S}_F$ for all faces. These are weak containment properties; when both spatial and facial objects are lines, equality is the corresponding strong property.

Lemma 3.2 (One-face reduction)**.** *If F is not equilateral, then*

$$\rho_F(\mathcal{S}_T) \subseteq \mathcal{S}_F \iff R \in \mathcal{S}_F, \tag{9}$$

$$\pi_F(\mathcal{S}_T) \subseteq \mathcal{S}_F \iff P \in \mathcal{S}_F \text{ and } R \in \mathcal{S}_F. \tag{10}$$

Consequently, OSCP *implies* CSCP.

Proof. Because $G = (3g + D)/4$, one has $\rho_F(G) = g$, proving (9). Affinity gives

$$\pi_F(G) = \frac{3g + P}{4}. \tag{11}$$

If $\lambda_D = (z + v + w)/(2L)$, then

$$W = \lambda_D D + (1 - \lambda_D)R, \quad \pi_F(W) = \lambda_D P + (1 - \lambda_D)R.$$

Since $0 < \lambda_D < 1$, these identities prove (10). □

Lemma 3.3 (Equilateral one-face reduction)**.** *If F is equilateral, so that $\mathcal{S}_F = \{g\}$, then*

$$\rho_F(\mathcal{S}_T) \subseteq \mathcal{S}_F \iff R = g,$$
$$\pi_F(\mathcal{S}_T) \subseteq \mathcal{S}_F \iff P = g \text{ and } R = g.$$

Proof. The central image always contains $\rho_F(G) = g$, so it is contained in the single point $\{g\}$ exactly when the image of W is also g. For orthogonal projection,

$$\pi_F(G) = \frac{3g + P}{4}, \qquad \pi_F(W) = \lambda_D P + (1 - \lambda_D)R,$$

with $0 < \lambda_D < 1$. Both projected points equal g exactly when $P = R = g$. □

In the Euler problem, orthogonal projection sends the spatial circumcenter to the facial circumcenter automatically. Here it does not send G to g; the altitude foot P intervenes. This is the source of the difference between the two classifications below.

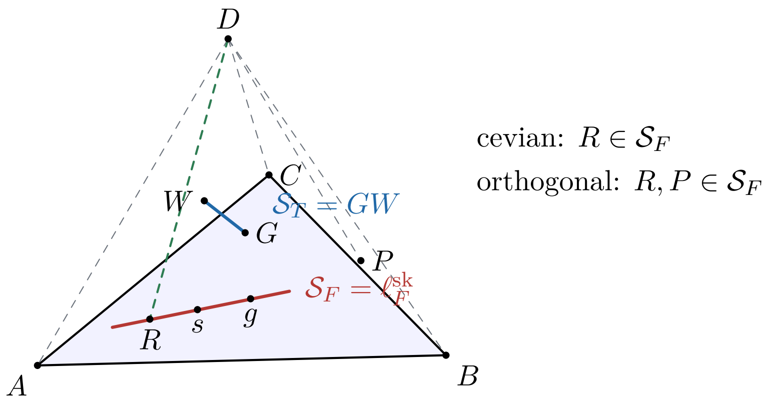

Figure 1: Cevian projection is governed by $R = DW \cap F$. Orthogonal projection requires the same condition and the independent altitude-foot condition $P \in \mathcal{S}_F$.

4 The universal jaw principle

The term *jaw* and the four-equal-cycle family were introduced by Hajja, Hammoudeh, Hayajneh, and Martini in their study of tetrahedral cevians [6]. The repeated appearance of jaws in the present projection problem is not special to GW, GO, or GI; it follows from a general symmetry mechanism.

Definition 4.1 (Natural equivariant center)**.** A *natural Euclidean center rule* assigns a unique point $Z(T)$ to each unlabeled tetrahedron and commutes with Euclidean isometries:

$$Z(\varphi T) = \varphi Z(T).$$

The analogous definition is used for a triangular center $z(F)$. A *central object* determined by two such rules is their joining line when the two points are distinct, and their common point when they coincide.

Theorem 4.2 (Universal jaw principle)**.** *Let $T = ABCD$ be a nondegenerate jaw labeled so that*

$$AB = BC = CD = DA.$$

Let K, N be the midpoints of AC, BD, and put

$$\ell = KN.$$

Then:

(i) *every natural equivariant tetrahedral center lies on ℓ;*

(ii) *on the face ABC, every natural equivariant triangular center lies on the isosceles reflection axis BK;*

(iii) *opposite-vertex cevian projection satisfies*

$$\rho_{ABC}(\ell) = BK;$$

(iv) *orthogonal projection satisfies*

$$\pi_{ABC}(\ell) \subseteq BK,$$

and in fact equality holds for every nondegenerate jaw.

The analogous statements hold cyclically for all four faces. Consequently, whenever two natural spatial center rules determine the full line ℓ and the corresponding two facial center rules determine the full reflection axis on each jaw face, both weak projection properties hold.

Proof. Write

$$A = K + p, \quad C = K - p, \qquad B = N + q, \quad D = N - q,$$

and let $t = N - K$. The four equal-edge equations imply

$$p \cdot q = p \cdot t = q \cdot t = 0.$$

For example, comparing AB^2 with BC^2, and AD^2 with CD^2, gives $p \cdot (t - q) = p \cdot (t + q) = 0$; comparing AB^2 with AD^2 then gives $q \cdot t = 0$. Thus rotation through π about ℓ sends

$$A \longleftrightarrow C, \qquad B \longleftrightarrow D$$

and preserves the tetrahedron. Its fixed-point set is exactly ℓ. If Z is a natural equivariant center, then

$$Z(T) = Z(\sigma T) = \sigma Z(T),$$

so $Z(T) \in \ell$. This proves (i).

The face ABC is isosceles with $AB = BC$. Reflection in BK interchanges A, C and fixes B; the same equivariance argument puts every natural face center on BK, proving (ii).

For cevian projection from D, the plane

$$\Pi = \operatorname{aff}(D, \ell)$$

contains K. It also contains B, because $N \in \ell$ and B, N, D are collinear. Hence

$$\Pi \cap \operatorname{aff}(ABC) = BK,$$

which proves (iii).

Finally, $\ell \perp AC$, while the normal to the plane ABC is also perpendicular to AC. Therefore the orthogonal projection of the direction of ℓ onto ABC remains perpendicular to AC. Because $K \in \ell \cap \operatorname{aff}(ABC)$, the projected object is either K or the unique line through K in the face perpendicular to AC, namely BK. The point alternative is impossible: in the notation above the direction t has scalar product $\|t\|^2 > 0$ with the in-face vector $B - K = t + q$, so it is not normal to the face. This proves (iv). Cyclic relabeling treats the other faces. $\square$

Remark 4.3. A generic jaw also has two reflection planes, perpendicular to its diagonals; their intersection is the half-turn axis. The qualification *natural equivariant* is necessary. An arbitrary labeled point rule need not respect the half-turn, and an arbitrary oblique projection need not send ℓ to a facial symmetry axis. The theorem covers the standard uniquely defined Euclidean centers, including the circumcenter, centroid, incenter, Monge point, unique Fermat-type centers, and the skeletal centroids.

There is, however, an equally necessary nondegeneracy qualification. If the two facial center rules coincide, their central object is a point, not the whole reflection axis, and symmetry alone does not force the projected spatial line into that point. This is precisely why jaws with an equilateral face require a separate boundary analysis below. For non-equilateral jaw faces the Euler line and the skeletal-centroid line are the full reflection axis, so the symmetry principle applies directly.

5 The cevian classification

From (3),

$$R \sim (x+y+z : x+u+v : y+u+w). \tag{12}$$

The line ℓ_F^{sk} is represented by barycentric rows $(1,1,1)$ and (u,y,x). Subtracting the Spieker row $(x+y, x+u, y+u)$, already on ℓ_F^{sk}, gives

$$R \in \ell_F^{\mathrm{sk}} \iff \det\begin{pmatrix} z & v & w \\ 1 & 1 & 1 \\ u & y & x \end{pmatrix} = 0.$$

The four face conditions are therefore

$$E_1 = (y-x)z + (x-u)v + (u-y)w = 0, \tag{13a}$$
$$E_2 = (z-x)y + (x-v)u + (v-z)w = 0, \tag{13b}$$
$$E_3 = (z-y)x + (y-w)u + (w-z)v = 0, \tag{13c}$$
$$E_4 = (v-u)x + (u-w)y + (w-v)z = 0. \tag{13d}$$

They satisfy $E_1 - E_2 + E_3 - E_4 = 0$.

Lemma 5.1 (The four-polynomial decomposition)**.** *The common zero set of* (13) *is the union*

$$\begin{aligned}
&\mathcal{C} : z+u = y+v = x+w, \\
&\mathcal{J}_1 : x = u = w = z, \qquad \mathcal{J}_2 :\ y = u = v = z, \qquad \mathcal{J}_3 :\ x = y = v = w, \\
&\mathcal{D} : x = w, \quad y = v, \quad z = u.
\end{aligned}$$

The statement is purely algebraic and is identical to the decomposition used in the Euler-projection companion after replacing each edge length here by its square there.

Proof. Put

$$\alpha = z+u-y-v, \qquad \beta = z+u-x-w.$$

Solving for v, w and substituting gives

$$\begin{aligned}
E_1 &= \alpha(u-x) + \beta(y-u), & E_2 &= \alpha(\beta+x-z) + \beta(y-u), \\
E_3 &= \alpha(\beta+x-u) + \beta(y-z), & E_4 &= \alpha(z-x) + \beta(y-z).
\end{aligned}$$

If $\alpha = \beta = 0$ one obtains $\mathcal{C}$. If exactly one of α, β vanishes, the equations give $\mathcal{J}_1$ or $\mathcal{J}_2$. If $\alpha\beta \neq 0$, then

$$E_2 - E_1 = -\alpha(w-x), \qquad E_1 - E_4 = (\alpha-\beta)(u-z),$$

so $w = x$ and either $\alpha = \beta$, which gives $\mathcal{J}_3$, or $u = z$, which then forces $y = v$ and gives $\mathcal{D}$. Direct substitution proves the converse inclusions. □

Definition 5.2. A tetrahedron is *circumscriptible*, or edge-incentric, if a sphere is tangent to all six edges. It is a *jaw* if its vertices can be cyclically labeled so that $AB = BC = CD = DA$, following [6]. It is *equifacial*, or a disphenoid, if opposite edges are pairwise equal.

The classical equivalent conditions for circumscriptibility are

$$z+u = y+v = x+w \tag{14}$$

and the existence of positive balloon radii with

$$AB = r_A + r_B, \quad AC = r_A + r_C, \quad \dots, \quad CD = r_C + r_D. \tag{15}$$

See Wu–Zhang [10] and Hajja [3].

Remark 5.3 (Power–1 interpretation)*.* The circumscriptible component $\mathcal{C}$ is exactly $\mathcal{P}_1$ of (companion P). Indeed $e_{ij} = p_i + p_j$, and on the face ABC

$$p_A = \frac{x+y-u}{2}, \qquad p_B = \frac{x+u-y}{2}, \qquad p_C = \frac{y+u-x}{2}.$$

The strict triangle inequalities make these three numbers positive, and the fourth parameter is positive by any lateral face. Thus the signed definition of $\mathcal{P}_k$ automatically collapses to the classical positive balloon parameters at $k = 1$.

Theorem 5.4 (Cevian classification)**.** *Let T be non-coplanar with no equilateral face. Then T has* CSCP *if and only if it is circumscriptible, a jaw, or equifacial. In labeled edge space, the components are*

$$\mathcal{C} : z + u = y + v = x + w, \tag{16}$$

$$\mathcal{J}_1 : x = u = w = z, \qquad \mathcal{J}_2 : \; y = u = v = z, \qquad \mathcal{J}_3 : \; x = y = v = w, \tag{17}$$

$$\mathcal{D} : x = w, \quad y = v, \quad z = u. \tag{18}$$

$\mathcal{C}$ has dimension four; each other component has dimension three.

Proof. By Lemma 3.2, the four face conditions are precisely (13). Lemma 5.1 gives their complete algebraic zero set. The component $\mathcal{C}$ is circumscriptible, equivalently $\mathcal{P}_1$, by (14) and Remark 5.3; the three $\mathcal{J}_i$ are the labeled jaw components; and $\mathcal{D}$ is equifacial. Conversely every listed component satisfies (13), so the one-face lemma gives CSCP. $\square$

On $\mathcal{D}$, all incident edge-length sums are equal, so $W = G$. Thus the equifacial branch enters as a weak, collapsed-line solution: cevian projection sends its point G to the face centroid g.

6 The orthogonal classification

The additional altitude-foot condition has an exact metric form.

Lemma 6.1 (Altitude foot on the skeletal facial line)**.** *Let $F = UVW$ be non-equilateral, with*

$$a = UV, \quad b = UW, \quad c = VW,$$

and let $XU = d, XV = e, XW = f$. Put

$$\tau = a^2 + b^2 - c^2, \quad A_0 = d^2 + a^2 - e^2, \quad B_0 = d^2 + b^2 - f^2, \quad \Delta = 4a^2b^2 - \tau^2.$$

The foot P of X on UVW lies on ℓ_F^{sk} if and only if

$$\begin{aligned}\Phi = \Delta(a-b) &+ (2b^2A_0 - \tau B_0)(c - 2a + b)\\ &+ (2a^2B_0 - \tau A_0)(2b - c - a) = 0.\end{aligned} \tag{19}$$

Proof. With U as origin and basis $V - U, W - U$, the Gram matrix is

$$K = \begin{pmatrix} a^2 & \tau/2 \\ \tau/2 & b^2 \end{pmatrix}.$$

If $P = (p_U : p_V : p_W)$ in normalized barycentrics, then

$$\begin{pmatrix} p_V \\ p_W \end{pmatrix} = K^{-1}\frac{1}{2}\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}, \qquad p_U = 1 - p_V - p_W.$$

The incenter has barycentrics $(c : b : a)$. Expanding

$$\det\begin{pmatrix} p_U & p_V & p_W \\ 1 & 1 & 1 \\ c & b & a \end{pmatrix} = 0$$

after multiplication by $4\det K = \Delta$ gives (19). $\square$

Lemma 6.2 (Component factorizations)**.** *For ABC opposite D, with the exact normalization in* (19)*:*

(i) *On the circumscriptible component,*

$$\Phi_{ABC} = 16r_D(r_A + r_B + r_C)(r_A - r_B)(r_A - r_C)(r_B - r_C). \tag{20}$$

(ii) *On the equifacial component, with opposite-edge values*

$$AB = CD = x, \quad AC = BD = y, \quad AD = BC = z,$$

$$\Phi_{ABC} = -4(x - y)(x - z)(y - z)(x + y + z)^2. \tag{21}$$

Proof. Substitute (15) into (19). Before changing to balloon radii, the nontrivial factor is

$$(x - u)(y - u)(x - y)(x + y - u - 2z).$$

Here

$$x - u = r_A - r_C, \quad y - u = r_A - r_B, \quad x - y = r_B - r_C, \quad x + y - u - 2z = -2r_D,$$

and direct expansion gives the positive prefactor $16(r_A + r_B + r_C)$ in (i). Substitution $w = x, v = y, u = z$ in (19) gives the exact factorization in (ii). □

Theorem 6.3 (Orthogonal classification)**.** *Let T be non-coplanar with no equilateral face. Then T has* OSCP *if and only if T is a jaw.*

Proof. Every jaw has OSCP. Label it so that $AB = BC = CD = DA$. The face ABC is isosceles with base AC, while D is equidistant from A, C. Hence the foot of D lies on the perpendicular bisector of AC, the common symmetry axis containing g, i, s. The same argument applies cyclically. A jaw also satisfies CSCP, so Lemma 3.2 proves sufficiency.

Conversely, OSCP ⇒ CSCP, so Theorem 5.4 reduces the proof to three components. On the circumscriptible component, (20) says that for the face opposite D, at least two of r_A, r_B, r_C coincide. Applying this to every face says that every three-subset of the four radii contains a repetition; hence the radii take at most two values. Three equal radii would make a face equilateral. Therefore the non-equilateral case has a $2 + 2$ split. For example,

$$r_A = r_C = p, \qquad r_B = r_D = q$$

gives

$$AB = BC = CD = DA = p + q,$$

a jaw.

On the equifacial component, (21) forces two of x, y, z to coincide. With opposite edges equal, this again makes a four-cycle of equal edges. The jaw components were already verified, so no other orthogonal solutions exist. □

7 Equilateral-face completion

If F is equilateral, then $g = i = s$ and the facial skeletal-centroid object is the single point $\mathcal{S}_F = \{g\}$. The polynomial equations (13) alone no longer encode the geometric condition, so this boundary must be treated directly.

Theorem 7.1 (Kite rigidity)**.** *Let T be a non-coplanar tetrahedron having an equilateral face. Then the following are equivalent:*

(i) *T has* CSCP*;*

(ii) *T has* OSCP*;*

(iii) *T is a tetrahedral kite: if ABC is an equilateral face, then $AD = BD = CD$.*

Every such kite is circumscriptible, hence belongs to $\mathcal{P}_1$.

Proof. Let the side of the equilateral face ABC be a. From (12),

$$R = \rho_{ABC}(W) \sim (2a + AD : 2a + BD : 2a + CD).$$

By Lemma 3.3, either all-face property requires $R = g = (1 : 1 : 1)$; hence

$$AD = BD = CD.$$

For the orthogonal property the same lemma also requires $P = g$, which follows automatically from the equality of the three lateral edges.

Conversely, suppose $AD = BD = CD$. The tetrahedron has a threefold rotation axis through D and the center g of ABC, and three reflection planes through that axis. The points G and W are fixed by the rotation and therefore lie on the axis. If the base and lateral edges have different lengths, then $G \neq W$ and $\mathcal{S}_T$ is exactly this axis. For a lateral face, its opposite base vertex and $\mathcal{S}_T$ lie in the relevant reflection plane; that plane meets the face in the symmetry axis ℓ_F^{sk}. This proves the central-projection condition. Orthogonal projection commutes with the same reflection and therefore also lands in the facial symmetry axis. On the equilateral base both images collapse to g. If the two edge lengths agree, the tetrahedron is regular and $G = W$; both images are then the corresponding face centroid directly. Thus both CSCP and OSCP hold in all cases.

Finally, with $x = y = u = a$ and $z = v = w = b$,

$$z + u = y + v = x + w = a + b,$$

so the kite is circumscriptible, equivalently in $\mathcal{P}_1$. □

Tetrahedra with a regular facet are called pre-kites in Hajja–Hayajneh–Hammoudeh [4]. Combining the theorem with the non-equilateral classifications gives the unrestricted result.

Corollary 7.2 (Complete classifications)**.** *For arbitrary non-coplanar tetrahedra:*

(i) *the* CSCP *locus is*

$$\mathcal{P}_1 \;\cup\; \{\textit{jaws with no equilateral face}\} \;\cup\; \{\textit{equifacial tetrahedra with no equilateral face}\};$$

(ii) *the* OSCP *locus is*

$$\{\textit{jaws with no equilateral face}\} \;\cup\; \{\textit{tetrahedral kites}\}.$$

8 Comparison with the Euler theorem

The central structural parallel is

$$\boxed{\begin{aligned} \text{squared-edge opposite sums} &\iff \mathcal{P}_2 \iff \text{orthocentricity}, \\ \text{edge-length opposite sums} &\iff \mathcal{P}_1 \iff \text{circumscriptibility}. \end{aligned}} \tag{22}$$

Thus the two projection theories select the $k = 2$ and $k = 1$ members of the power hierarchy introduced in (companion P). The same residual components, jaws and equifacial tetrahedra, occur because the decomposition uses the same four formal polynomials.

Table 1: Euler and skeletal-centroid classifications when no face is equilateral. Dimensions include scale.

Feature	Euler object	Skeletal-centroid object
Spatial object	$\mathrm{aff}\{G, O\}$ (equivalently the Monge line)	$\mathrm{aff}\{G, W\}$
Facial object	$\mathrm{aff}\{g, o\} = \mathrm{aff}\{g, h\}$	$\ell_F^{\mathrm{sk}} = X(2)X(10)$
Power–k member	$\mathcal{P}_2$	$\mathcal{P}_1$
Basic variables	squared lengths	lengths
Maximal component	orthocentric, dimension 4	circumscriptible, dimension 4, cevian only
Orthogonal locus	orthocentric ∪ jaws ∪ equifacial	jaws
Cevian locus	orthocentric ∪ jaws ∪ equifacial ∪ diametral mirrors	circumscriptible ∪ jaws ∪ equifacial
Canonical anchor	$\pi_F(O) = o$	$\rho_F(G) = g$

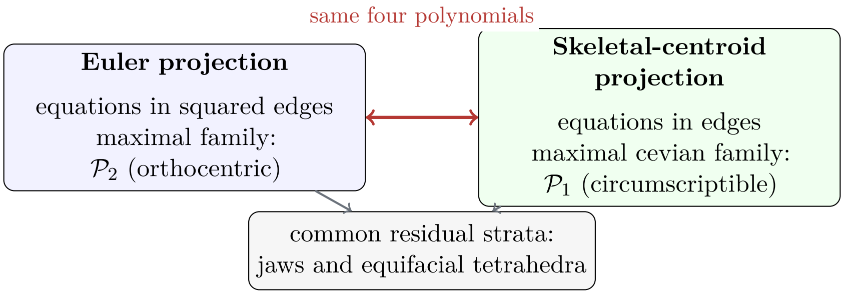

Figure 2: The algebraic bridge between the two projection theories.

The projection mechanisms differ. In Euler geometry, $\pi_F(O) = o$, while cevian projection acquires an exceptional branch when $O \in \mathrm{aff}(F)$, producing diametral mirrors. In the present geometry, $\rho_F(G) = g$ is the canonical anchor, but

$$\pi_F(G) = \frac{3g + P}{4}.$$

The extra condition $P \in \ell_F^{\mathrm{sk}}$ cuts the four-dimensional circumscriptible family down to jaws.

The dimension-theoretic companion (companion D) concerns Z-cevian concurrence rather than projection, so its nonzero-weight analytic no-5 obstruction does not imply the present classification. Nevertheless the four papers display a consistent stratification: universal phenomena live in dimension six, the power families $\mathcal{P}_1, \mathcal{P}_2$ in dimension four, and the recurring jaw geometry in dimension three.

9 Examples

Example 9.1 (Cevian but not orthogonal)**.** Let

$$A = (0, 0, 0), \quad B = (3, 0, 0), \quad C = (0, 4, 0), \quad D = \left(-\frac{1}{3}, -1, \frac{\sqrt{215}}{3}\right).$$

The edges are

$$(x, y, z, u, v, w) = (3, 4, 5, 5, 6, 7),$$

with balloon radii $(r_A, r_B, r_C, r_D) = (1, 2, 3, 4)$. Thus

$$z + u = y + v = x + w = 10.$$

The tetrahedron is circumscriptible and has CSCP. Its four balloon radii are distinct, so (20) fails; it has no OSCP.

Example 9.2 (A jaw with both properties)**.** For

$$A = (-1, 0, 0), \quad B = (0, 2, 1), \quad C = (1, 0, 0), \quad D = (0, -2, 1)$$

one has

$$AB = BC = CD = DA = \sqrt{6}, \qquad AC = 2, \qquad BD = 4.$$

Its half-turn axis joins the midpoints of AC, BD, contains G, W, and projects to the symmetry axis ℓ_F^{sk} of every isosceles face.

Example 9.3 (A weak equifacial solution)**.** Take

$$\begin{aligned} A &= (1, 2, 3), & B &= (1, -2, -3), \\ C &= (-1, 2, -3), & D &= (-1, -2, 3). \end{aligned}$$

The opposite edge pairs have lengths

$$(x, y, z, u, v, w) = (\sqrt{52}, \sqrt{40}, \sqrt{20}, \sqrt{20}, \sqrt{40}, \sqrt{52}).$$

Here $W = G$, so the cevian object is a point mapping to every face centroid. Since the three opposite-edge values are distinct, (21) shows that the orthogonal property fails.

10 Dimensions and interpretation

Table 2: Principal components for non-equilateral faces.

Family	Edge condition	Dim.	CSCP	OSCP
Circumscriptible	$z + u = y + v = x + w$	4	✓	only jaw strata
Jaw	one four-cycle of equal edges	3	✓	✓
Equifacial	$x = w, \ y = v, \ z = u$	3	✓	only jaw strata
General	six realizable lengths	6	–	–

In balloon coordinates, the orthogonal cut has a simple combinatorial meaning: every triple of four radii must repeat a value. The non-equilateral solutions are the $2 + 2$ colorings of the vertices by two radii. Their cross-edges form a four-cycle of equal length, precisely a jaw.

A generic circumscriptible tetrahedron has no Euclidean symmetry and survives only the cevian test. A jaw has a half-turn symmetry forcing its spatial mass line and facial lines into compatible axes. An equifacial tetrahedron has more symmetry, but its line has collapsed; cevian projection respects the collapsed centroid, whereas orthogonal projection moves it to a face circumcenter and generally breaks compatibility with ℓ_F^{sk}.

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11 Conclusion and open directions

The ordinary-edge and squared-edge systems explain the relation between the skeletal and Euler classifications. The mass-center identities and classical families are background; the all-face projection classifications, including collapsed facial objects, are the results established here.

Natural next questions include:

1. classify strong versions, excluding collapsed spatial or projected objects;
2. determine the remaining area-radical components of the cevian and orthogonal centroid–incenter projection loci;
3. replace GW by WQ, whose facial counterpart is again $X(2)X(10)$;
4. study projection properties of the principal axes of the three mass distributions;
5. extend the construction to intermediate skeletal centroids of n-simplices using the formulas of Krantz–McCarthy–Parks [9].

Proof transparency. The mass-center formulas, four-polynomial decomposition, and component factorizations needed for the classifications are all displayed in the text. No computer-algebra calculation or external Mathematica notebook is used in a proof, and the bundle supplies optional independent exact checks.

A The centroid–incenter problem

The surface centroid suggests a second problem. Propositions 2.1 and 2.2 give the line identities

$$\ell_T^{GI} = \operatorname{aff}\{G, Q\} = \operatorname{aff}\{G, I\}, \qquad \ell_F^{\mathrm{sk}} = \operatorname{aff}\{g, s\} = \operatorname{aff}\{g, i\},$$

with the affine ratios

$$Q - G = \frac{G - I}{3}, \qquad s - g = \frac{g - i}{2}.$$

The incenter has no simple projection law analogous to $\pi_F(O) = o$ or $\rho_F(G) = g$. For $F = ABC$,

$$I \sim (\Delta_A : \Delta_B : \Delta_C : \Delta_D),$$

so

$$\rho_F(I) \sim (\Delta_A : \Delta_B : \Delta_C). \tag{23}$$

The cevian condition for the line ℓ_T^{GI} is

$$\det \begin{pmatrix} \Delta_A & \Delta_B & \Delta_C \\ 1 & 1 & 1 \\ u & y & x \end{pmatrix} = 0. \tag{24}$$

The face areas are Heron radicals in the six edge lengths, so this does not reduce to the bilinear system (13).

Orthogonal projection is stricter. If $P = (p_A : p_B : p_C)$ in normalized face barycentrics, then

$$\pi_F(I) = \frac{(\Delta_A + \Delta_D p_A)A + (\Delta_B + \Delta_D p_B)B + (\Delta_C + \Delta_D p_C)C}{\Sigma}.$$

Thus one needs both $P \in \ell_F^{\mathrm{sk}}$ and

$$\det \begin{pmatrix} \Delta_A + \Delta_D p_A & \Delta_B + \Delta_D p_B & \Delta_C + \Delta_D p_C \\ 1 & 1 & 1 \\ u & y & x \end{pmatrix} = 0.$$

Even after the altitude-foot condition, a second nonlinear area condition remains.

Relevant center-coincidence results were developed by Edmonds, Hajja, and Martini [1]; equifacial tetrahedra and their coincident centers were treated by Hajja and Walker [5]. For the present projection question:

(i) Every jaw with no equilateral face has both weak ℓ_T^{GI}-projection properties. Half-turn symmetry fixes the spatial line, while the nondegenerate facial object ℓ_F^{sk} is the full isosceles symmetry axis.

(ii) Every equifacial tetrahedron has the weak cevian property because $G = I$ and $\rho_F(G) = g$.

(iii) A generic scalene equifacial tetrahedron fails orthogonally. Here $G = I = O$, so its orthogonal image is the face circumcenter o, which is not generally on ℓ_F^{sk}. Formula (21) shows that within the equifacial family the altitude-foot condition holds exactly on jaw strata.

The non-equilateral qualification in (i) is essential. For example, take

$$A = (-\tfrac{1}{2}, 0, 0), \quad C = (\tfrac{1}{2}, 0, 0), \quad B = (0, \tfrac{\sqrt{3}}{4}, \tfrac{3}{4}), \quad D = (0, -\tfrac{\sqrt{3}}{4}, \tfrac{3}{4}).$$

Then

$$AB = BC = CD = DA = AC = 1, \qquad BD = \frac{\sqrt{3}}{2}.$$

Thus the tetrahedron is a jaw and ABC is equilateral. Directly,

$$G = (0, 0, \tfrac{3}{8}), \qquad I = (0, 0, 4 - \sqrt{13}),$$

so ℓ_T^{GI} is the nondegenerate jaw axis. Here $\rho_{ABC}(G) = g$, but $\rho_{ABC}(I) \neq g$, since the median Dg meets the z-axis only at G. Thus the projected affine object is the full reflection axis of ABC, whereas the facial object is the single point $\{g\}$; the weak central property fails. Equilateral-face cases therefore require a separate analysis and are not classified here.

Computational materials

The accompanying bundle contains editable sources, build instructions, independent exact-check scripts and their recorded outputs. These support the displayed calculations; open global classification problems are not certified by local checks.

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